Shape Control of Cubic B-spline and NURBS Curves by Knot Modifications

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Abstract

Shape control methods of cubic B-spline and NURBS curves by the modification of their knot values, and simultaneous modification of weights and knots are presented. Theoretical aspects of knot modification concerning the paths of points of a curve and the existence of an envelope for the family of curves resulted by a knot modification are also discussed for curves of degree k.

1. Introduction

B-spline and NURBS curves are standard description methods of CAD systems and widely used in computer aided design today. There are several books and papers on these curves describing their properties, with the help of which one can apply them as powerful design tools.

A k^{th} degree B-spline curve is uniquely defined by its control points and knot values, while in terms of NURBS curves the weight vector has to be specified in addition. The shape modification of these curves plays central role in CAD, hence numerous methods have been presented to control the shape of a curve by modifying one of its data mentioned above. The most basic possibilities can be found in any book of the field (e.g. in [6]). Further control point-based shape modification is discussed in [2] and [5], weight-based modification is described e.g. in [3] and [5], while others present shape control by simultaneous modification of control points and weights (see [1], [7]).

It is also well-known that the change of the knot vector affects the shape of the curve. The properties of this change, however, have not been described yet. The aim of this paper is to present the geometrical and mathematical representation of the effects of knot modification for B-spline curves. After the basic definitions some theoretical results are presented, by means of which one can describe the effects of the modification of a knot value on the shape of the curve. In the next sections constrained based shape control possibilities are discussed modifying knot values of a non-rational B-spline curve, while the effect of simultaneous modification of knots and weights is presented in the rational case. In these latter sections we restrict our consideration to curves of degree 3, since this is the most widely used type of B-spline and NURBS curves.

2. Theoretical results

In this section the modification of a knot value of a k^{th} degree B-spline curve will be examined. We begin our discussion with the basic definitions.

Definition 1 *The recursive function $N_j^k(u)$ given by the equations*

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}), \\ 0 & \text{otherwise} \end{cases}$$
$$N_j^k(u) = \frac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \frac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u)$$

is called normalized B-spline basis function of order k (degree $k-1$). The numbers $u_j \leq u_{j+1} \in \mathbb{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

Definition 2 *The curve $\mathbf{s}(u)$ defined by*

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\, \mathbf{d}_l \,, u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order k (degree $k-1$), where $N_l^k(u)$ is the l^{th} normalized B-spline basis function, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. The points $\mathbf{d}_i$ are called control points or de Boor-points, while the polygon formed by these points is called control polygon.

The j^{th} span of the B-spline curve can be written as

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \ u \in [u_j, u_{j+1}).$$

Modifying the knot u_i, the point of this span associated with the fixed parameter value $\tilde{u} \in [u_j, u_{j+1})$ will move along the curve

$$\mathbf{s}_j(\tilde{u}, u_i) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(\tilde{u}, u_i), u_i \in [u_{i-1}, u_{i+1}].$$

Hereafter, we refer to this curve as the *path* of the point $\mathbf{s}_j(\tilde{u})$. In [4] the authors proved the following basic properties of these paths:

Theorem 1 *Modifying the knot value* $u_i \in [u_{i-1}, u_{i+1}]$ *of a* k^{th} *order B-spline curve, the points of the spans* $\mathbf{s}_{i-k+1}(u), ..., \mathbf{s}_{i+k-2}(u)$ *moves along rational curves. The degree of these paths decreases symmetrically from* $k-1$ *to* 1 *as the indices of the spans getting farther from i, i.e. the paths* $\mathbf{s}_{i-m}(\tilde{u}, u_i)$ *and* $\mathbf{s}_{i+m-1}(\tilde{u}, u_i)$ *are rational curves of degree* $k-m$ *with respect to* u_i, $(m = 1, \ldots, k-1)$.

The theorem states that modifying $u_i \in [u_{i-1}, u_{i+1}]$ the points of the spans $\mathbf{s}_{i-k+1}(\tilde{u}, u_i)$ and $\mathbf{s}_{i+k-2}(\tilde{u}, u_i)$ move along straight lines. One can easily prove the following corollary, which will be strongly used in the next section.

Corollary 1 *If* u_i *runs from* u_{i-1} *to* u_{i+1}, *then the points of the span* $\mathbf{s}_{i-k+1}(\tilde{u}, u_i)$ *and* $\mathbf{s}_{i+k-2}(\tilde{u}, u_i)$ *move along straight lines parallel to the side* $\mathbf{d}_{i-k}, \mathbf{d}_{i-k+1}$ *and* $\mathbf{d}_{i-1}, \mathbf{d}_i$ *of the control polygon, respectively.*

Beside these paths we can also consider the one-parameter family of B-spline curves

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \quad \begin{array}{l} u \in [u_{k-1}, u_{n+1}] \\ u_i \in [u_{i-1}, u_{i+1}) \end{array}$$

which is resulted by the modification of the knot value u_i.

In case of $k = 3$ the spans of the curves are parabolic arcs. It is a well-known fact, that the tangent lines of these arcs at the knot values coincide with the corresponding sides of the control polygon. Modifying a knot value u_i the tangent line remains the same, which can be interpreted as the side of the control polygon is an envelope of the family of these quadratic B-spline curves. The generalization of this property has also been proved by the authors for arbitrary k (see [4]):

Theorem 2 *The family of the* k^{th} *order B-spline curves* $\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i)$, $(k > 2)$ *has an envelope which is also a B-spline curve of order* $(k-1)$ *and can be written in the form*

$$\mathbf{b}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l N_l^{k-1}(v), \ v \in [v_{i-1}, v_i],$$

where $v_j = \begin{cases} u_j & \text{if } j < i \\ u_{j+1} & \text{if } j \geq i \end{cases}$, *that is the* i^{th} *knot value is removed from the knot vector* $\{u_j\}$ *of the original curves.*

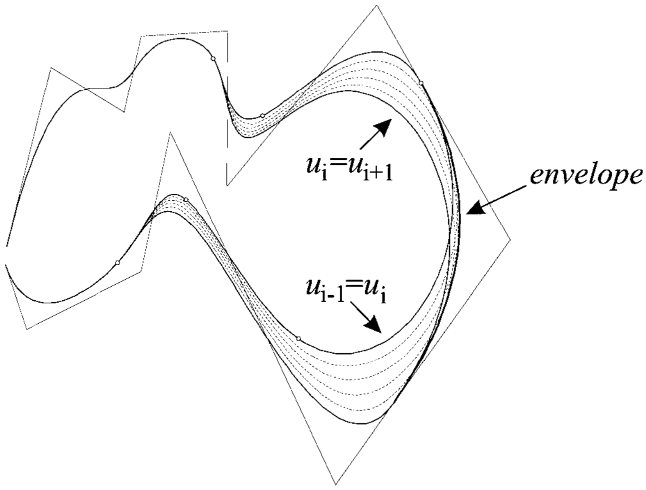

Figure 1. The envelope of the family of cubic B-spline curves is a quadratic B-spline curve with the same control polygon.

Until now only non-rational B-spline curves have been examined, but similar results hold for the rational case. A rational B-spline curve can always be considered as a central projection of a non-rational B-spline curve. The degree of a curve cannot increase by a central projection, thus Theorem 1 and its corollary hold for paths of the points of a NURBS curve, except the parallel paths will be concurrent, which will be discussed in the next section. Similarly, Theorem 2 holds for the rational case too, but the envelope will also be a NURBS curve. Fig.1 shows an envelope of the family of cubic B-spline curves resulted by the modification of one of the knot values.

These theoretical results help us to develop some interesting tools for shape control of B-spline and NURBS curves by the modification of their knot values, that will be examined in the next section.

3. Shape control

For the sake of simplicity we restrict our consideration for the case $k = 4$. Cubic curves are powerful design tools for most of the applications in the plane as well as in the 3D space. Some of the algorithms discussed below can be generalized for arbitrary k, while others use the specific properties of cubic curves.

3.1. Non-rational B-spline curve passing through a point

Let $\mathbf{s}(u)$ be a non-rational cubic B-spline curve with control points $\mathbf{d}_i, (i = 0, ..., n)$ and knot values $u_j, (j = 0, ..., n+4)$. Until now the only possibility for the modification of this curve has been the repositioning of its control points. Now we give an algorithm for changing this curve by modifying its knot values in such a way that the curve will pass through a given point $\mathbf{p}$ at the given parameter value $\tilde{u}$. This point, of course cannot be anywhere: the algorithm works if this point is inside the region defined by the sides of the control polygon and the envelopes mentioned in Theorem 2, which are parabolic arcs in the cubic case.

Let the point $\mathbf{p}$ be in the region defined by the control points $\mathbf{d}_{j-2}, \mathbf{d}_{j-1}, \mathbf{d}_j$. Let a parameter value $\tilde{u} \in [u_j, u_{j+2})$ be also given. Consider a quadratic B-spline curve $\mathbf{b}(v)$ with the same control points and the knot values $v_0 = u_0, ...v_{j-1} = u_{j-1}$, $v_j = u_j$, $v_{j+1} = u_{j+2}, ..., v_{n+3} = u_{n+4}$. Hence the given value $\tilde{u} \in [v_j, v_{j+1})$. Consider the j^{th} span of the quadratic curve

$$\mathbf{b}_j(v) = \sum_{l=j-2}^{j} N_l^3(v)\,\mathbf{d}_l, \quad v \in [v_j, v_{j+1}).$$

Using the monotonicity of the knot values one can write

$$\begin{aligned} v - v_{j-1} &= (v_{j+1} - v_{j-1}) - (v_{j+1} - v) \\ v_{j+2} - v &= (v_{j+2} - v_j) - (v - v_j). \end{aligned}$$

Substituting these formulae to the original equation we obtain the form

$$\begin{aligned} \mathbf{b}_j(v) &= \mathbf{d}_{j-1} + N_{j-2}^3(v)\,(\mathbf{d}_{j-2} - \mathbf{d}_{j-1}) \\ &\quad + N_j^3(v)\,(\mathbf{d}_j - \mathbf{d}_{j-1}). \end{aligned}$$

Now consider the affine coordinate system the origin of which is $\mathbf{d}_{j-1}$ and the base vectors are $\mathbf{e}_1 = \mathbf{d}_{j-2} - \mathbf{d}_{j-1}$ and $\mathbf{e}_2 = \mathbf{d}_j - \mathbf{d}_{j-1}$. Let the coordinates of the given point $\mathbf{p}$ in this coordinate system be x and y. This yields the following system of equations:

$$\begin{aligned} \frac{(v_{j+1} - v)(v_{j+1} - v)}{(v_{j+1} - v_{j-1})(v_{j+1} - v_j)} &= x \\ \frac{(v - v_j)(v - v_j)}{(v_{j+2} - v_j)(v_{j+1} - v_j)} &= y \end{aligned}$$

Hence x, y and $v = \tilde{u}$ are given, one can choose two unknowns from the knot values $(v_{j-1}, v_j, v_{j+1}, v_{j+2})$. The system can be solved for any two unknowns, but to avoid the unnecessary changes of farther spans it is better to chose two neighboring values, thus 8 spans will be modified. Solving the system e.g. for v_{j-1}, v_j and considering the quadratic curve $\overline{\mathbf{b}}(v)$ with these knot values $\overline{\mathbf{b}}(\tilde{u}) = \mathbf{p}$ holds. Therefore, because of Theorem 2, the cubic curve $\overline{\mathbf{s}}(u)$ with the knot values $(...u_{j-1} = v_{j-1}$, $u_j = v_j$, $u_{j+1} = \tilde{u}$, $u_{j+2} = v_{j+1}...)$ also passes through the point $\mathbf{p}$ at the parameter value $\tilde{u}$.

Since we have four free parameters v_{j-1}, v_j, v_{j+1} and v_{j+2}, some additional conditions can be assumed for the quadratic curve $\overline{\mathbf{b}}(v)$ in advance. Such a condition can be the given tangent direction at $\mathbf{p}$, but the initial position of $\mathbf{p}$ and the given direction cannot be arbitrary. To describe the permissible positions and directions, consider parabolic arcs having the tangents $\mathbf{d}_{j-2}, \mathbf{d}_{j-1}$ and $\mathbf{d}_j, \mathbf{d}_{j-1}$. The tangent points are $\mathbf{b}(v_j)$ and $\mathbf{b}(v_{j+1})$, i.e. the points where the spline arcs are connected. The extreme positions of the points $\mathbf{b}(v_j)$ and $\mathbf{b}(v_{j+1})$ of the parabolic arc are $\mathbf{d}_{j-2}$ and $\mathbf{d}_j$,respectively. If the position of both end-points are extreme, then the parabolic arc defined by the control points $\mathbf{d}_{j-2}, \mathbf{d}_{j-1}, \mathbf{d}_j$ will be obtained, hence the point $\mathbf{p}$ can be given in the region defined by this arc and the two sides ($\mathbf{d}_{j-2}, \mathbf{d}_{j-1}$ and $\mathbf{d}_j, \mathbf{d}_{j-1}$) of the control polygon (see the grey area in Fig. 2).

If the point $\mathbf{p}$ is on this arc, then the tangent direction cannot be given in advance, since a parabolic arc is uniquely defined by two of its points and the tangents in them. However if the point $\mathbf{p}$ is an inner point of the area mentioned above, then the tangent line can be given in addition.

The extreme positions of this tangent line are given by the tangents of the extreme parabolic arcs passing through the point $\mathbf{p}$ and fulfilling the conditions. To obtain these extreme arcs, consider the following two situations: $\mathbf{b}(v_j) = \mathbf{d}_{j-2}$ and $\mathbf{b}(v_{j+1})$ is an inner point of the segment $\mathbf{d}_{j-1}, \mathbf{d}_j$, or $\mathbf{b}(v_{j+1}) = \mathbf{d}_j$ and $\mathbf{b}(v_j)$ is an inner point of the segment $\mathbf{d}_{j-2}, \mathbf{d}_{j-1}$. These parabolic arcs can easily be calculated by considering the affine coordinate system described above in this section, in which let the coordinates of the point $\mathbf{p}$ be (x, y). The control points of the first extreme arc are: $\mathbf{d}_{j-2}, \mathbf{d}_{j-1}, \mathbf{d}_{j-1} + \mu\mathbf{e}_2$, $\mu < 1$, and it can be written in the parametric form

$$\mathbf{c}(v) = \mathbf{d}_{j-1} + (1-v)^2\,\mathbf{e}_1 + v^2\mu\mathbf{e}_2.$$

One of its points will be the point $\mathbf{p}$ at the parameter value $v_0 \in (0, 1)$. For this point

$$\mathbf{p} = \mathbf{d}_{j-1} + x\mathbf{e}_1 + y\mathbf{e}_2$$

holds. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent, hence from the equation $\mathbf{p} = \mathbf{c}(v_0)$ we obtain the solutions $v_0 = 1 - \sqrt{x}$, $\mu = y/(1-\sqrt{x})^2$. Fig. 2 shows the two extreme parabolic arcs passing through $\mathbf{p}$ and their tan-

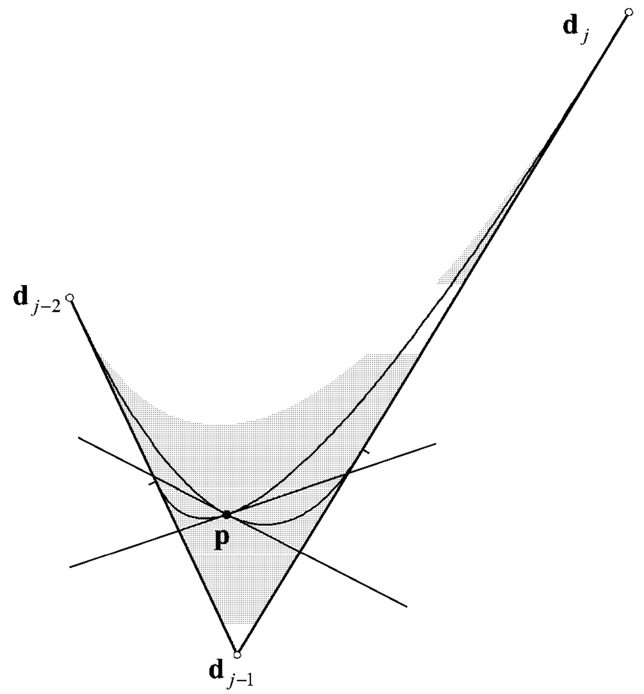

Figure 2. The permissible positions of the points p and the tangent directions of the parabolic arc .

gent lines at $\mathbf{p}$. The tangent direction can be given in this angular domain.

3.2. NURBS curve passing through a point

It is a well-known fact, that the modification of the weight w_j of a NURBS curve causes a perspective functional translation of points of the effected arcs, i.e. it pulls/pushes points of the curve toward/away from the control point $\mathbf{d}_j$. If a given point is on one of the line segments of the paths of this perspective change, one can easily compute the new weight value in such a way, that the new curve will pass through the given point. This point can *almost* be anywhere in the convex hull, but for $k > 3$ these concurrent line segments starting from $\mathbf{d}_j$ do not sweep the entire area of the triangle $\mathbf{d}_{j-1}, \mathbf{d}_j, \mathbf{d}_{j+1}$, cf. the gray area in Fig. 3. If the given point is close to the side of the control polygon, i.e. it is out of the shaded region of Fig. 3, the problem can only be solved with the change of two neighboring weights. Now, we give an algorithm to solve this problem with the change of one weight and one knot value.

Let $\mathbf{s}(u)$ be a cubic NURBS curve and $\mathbf{p}$ a point in the triangle $\mathbf{d}_{j-1}, \mathbf{d}_j, \mathbf{d}_{j+1}$. Consider the quadratic envelope $\mathbf{b}(v)$ of the family of NURBS curves $\mathbf{s}(u, u_{j+1})$. This parabolic arc intersects all the lines starting from $\mathbf{d}_j$ in this triangle, thus suitably changing the weight w_j there will be a parameter value $\widetilde{v}$, for which $\mathbf{b}(\widetilde{v}) = \mathbf{p}$. If we modify the knot value $u_{j+1}$of the cubic curve to be $u_{j+1} = \widetilde{v}$, the cubic curve will also pass through the point $\mathbf{p}$. This type of shape modification is illustrated in Fig. 3.

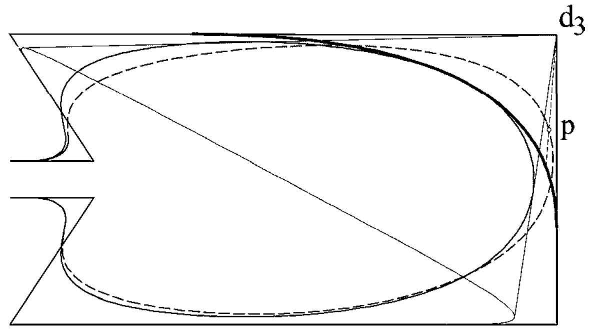

Figure 3. Modifying the weight w_3 and the knot u_4, the NURBS curve passes through the given point p which is outside the area accessible by the modification of w_3 only.

In this subsection the quadratic envelope has been modified by a weight, where the points of the curve moves along straight lines toward a control point. Similar effect, however, can be achieved in terms of non-rational quadratic B-spline curves by appropriate simultaneous modification of two knot values. More precisely, from the definition of the B-spline functions and the Corollary of Theorem 1 one can easily prove the following property:

Theorem 3 *Consider the quadratic non-rational B-spline curve $\mathbf{s}(u)$, and simultaneously modify its knots u_i and u_{i+3} in an equal manner, i.e. let $u_i = u_i + \lambda$, $u_{i+3} = u_{i+3} - \lambda$. As a result of this modification, points of the span $\mathbf{s}_{i+1}(u)$ move along concurrent straight lines, if and only if,*

$$u_{i+2} - u_i = u_{i+3} - u_{i+1}$$

holds. The common point of these straight lines is $\mathbf{d}_i$ (see Fig. 4).

Proof. As we have seen above, the span $\mathbf{s}_{i+1}(u)$ can be written in the form

$$\mathbf{s}_{i+1}(u) = \mathbf{d}_i + N_{i-1}^3(\mathbf{d}_{i-1} - \mathbf{d}_i) + N_{i+1}^3(\mathbf{d}_{i+1} - \mathbf{d}_i).$$

Applying the knot modification of the theorem, we obtain the family of curves

$$\begin{aligned} \mathbf{s}_{i+1}(u, \lambda) \quad = \quad & \mathbf{d}_i + \frac{u_{i+2} - u}{u_{i+2} - u_i - \lambda} N_i^2(\mathbf{d}_{i-1} - \mathbf{d}_i) + \\ & \frac{u - u_{i+1}}{u_{i+3} - \lambda - u_{i+1}} N_{i+1}^2(\mathbf{d}_{i+1} - \mathbf{d}_i) \end{aligned} \tag{1}$$

Assuming the equality $\delta = u_{i+2} - u_i = u_{i+3} - u_{i+1}$, we can factor out $1/(\delta - \lambda)$ and we obtain

$$\mathbf{s}_{i+1}(u,\lambda) = \mathbf{d}_i + \frac{1}{\delta - \lambda}\left((u_{i+2} - u)\, N_i^2(\mathbf{d}_{i-1} - \mathbf{d}_i) + (u - u_{i+1})\, N_{i+1}^2(\mathbf{d}_{i+1} - \mathbf{d}_i)\right)$$

which is a family of straight line segments and the pencil of lines determined by them has the centre $\mathbf{d}_i$.

Conversely, if $u_{i+2} - u_i \neq u_{i+3} - u_{i+1}$ then the rational curves (1) have two points at infinity (at $\lambda = u_{i+2} - u_i$ and $\lambda = u_{i+3} - u_{i+1}$), therefore they can not be straight lines. ■

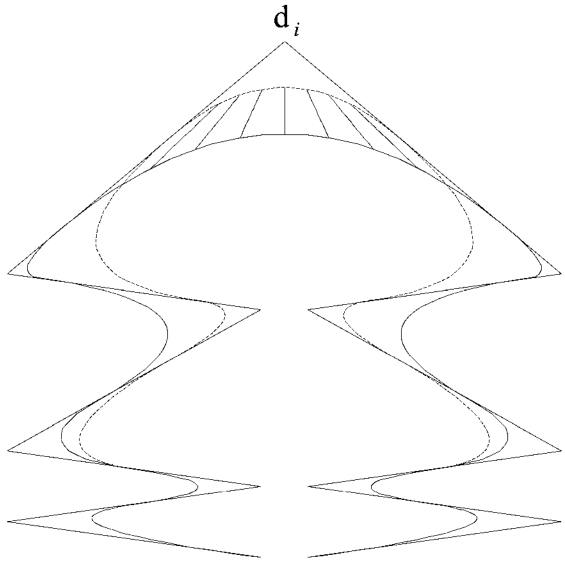

Figure 4. Simultaneous modification of two knot values yields a perspective change of the span of a non-rational quadratic B-spline curve.

The modification of these two knot values, of course, is not so effective, than that of a weight, because the region of change is greater in the latter case while the number of changing spans is fewer (7 for the two knot values and 3 for the weight), but we have to emphasize, that this theorem allows us to modify non-rational B-spline curves similarly to NURBS curves.

3.3. Modification of two weights and a knot value of a NURBS curve

Modifying two neighboring weights w_j, w_{j+1} of a NURBS curve the points of the curve move along straight lines toward or away from the leg $\mathbf{d}_j, \mathbf{d}_{j+1}$ of the control polygon. This change is neither perspective nor parallel. This property can be made more intuitive geometrically by modifying a knot value in addition. Thus the points of a span of the curve will move along concurrent lines passing through any given point of the line $\mathbf{d}_j, \mathbf{d}_{j+1}$ except the inner point of the leg. As we have mentioned in the preceding section, modifying a knot value u_j of a cubic NURBS curve the points of the spans $\mathbf{s}_{j-3}, \mathbf{s}_{j+2}$ will move along two families of concurrent straight lines. Considering the span $\mathbf{s}_{j-3}$ and assuming that $w_{j-4} \neq w_{j-3}$ the following result can be achieved: modifying the knot value u_j the points of this span move along concurrent lines the centre of which is on the line $\mathbf{d}_j, \mathbf{d}_{j+1}$ and its barycentric coordinates are

$$\left(\frac{w_{j-4}}{w_{j-4} - w_{j-3}}, 1 - \frac{w_{j-4}}{w_{j-4} - w_{j-3}}\right).$$

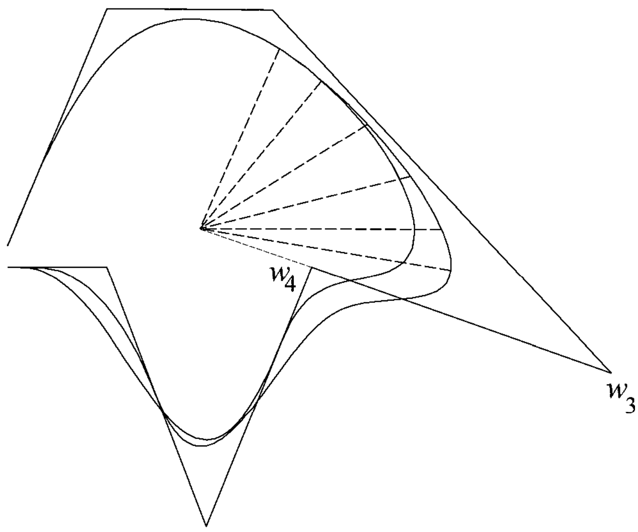

Figure 5. Modifying the knot value u_7, points of the span $\mathbf{s}_4$ moves along concurrent straight lines the centre of which depends on w_3 and w_4 and can be arbitrary located on the line of $\mathbf{d}_3, \mathbf{d}_4$, except the inner points of the segment.

One can easily see, that one of its coordinates must be negative with the usual assumption $w_j \geq 0, \forall j$. Hence this centre cannot be on the leg $\mathbf{d}_j, \mathbf{d}_{j+1}$ but on the rest of the line. Fig. 5 shows a case of this type of modification.

4. Conclusions and further research

This paper has been devoted to the shape control of cubic B-spline and NURBS curves. These curves can uniquely be defined by their degree, control points, weights and knot vector. While the effect of the modification of the preceding data has been widely published and used, the change of the knot vector has not been studied yet. At the first section some theoretical results have been presented in terms of the paths of the points of the curve when one of its knot values is modified, and the existence of an envelope of the resulted family of curves. Applying these results some shape control methods have been presented in the next section. Modifying one or some of the knot values of a non-rational B-spline one can achieve constraint-based modification, such as obtaining a curve passing through a given point, or a shape modification which is similar to the effect of the modification of a weight in the rational case. For NURBS curves simultaneous change of one or two weights and knot values has been presented, the result of which is a NURBS curve passing through a given point or a geometrically simple perspective shape modification.

Our objective in our further research, besides the knot-based constrained shape modification of curves of arbitrary order k, is to study the theoretical aspects of knot modifications for surfaces, which will hopefully generate some shape control methods both for B-spline and NURBS surfaces.

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