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The Effect of Knot Modifications on the Shape of B-spline Curves

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Abstract. This paper is devoted to the shape control of B-spline curves achieved by the modification of one of its knot values. At first those curves are described along which the points of a B-spline curve move under the modification of a knot value. Then we show that the one-parameter family of k^{th} order B-spline curves obtained by modifying a knot value has an envelope which is also a B-spline curve of order $k-1$.

Key Words: B-spline curves

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1. Introduction

B-spline and NURBS curves are standard description methods and hence widely used in Computer Aided Design today. There are several books and papers on these curves describing their properties, with the help of which one can apply them as powerful design tools.

A *B-spline curve* is uniquely defined by its degree, control points and knot values, while in terms of *NURBS curves* the weight vector has to be specified in addition. The modification of a curve plays a central role in CAD systems, hence numerous methods are presented to control the shape of a curve by modifying one of its data mentioned above. The most basic possibilities can be found in any book of the field. Further control point-based shape modification is discussed in [5] and [3], weight-based modification is described e.g. in [5] and [4], while others present shape control by simultaneous modification of control points and weights (see [7], [1]).

The effect of a change of the knot vector on the shape of the curve, however, has not been described yet. Even in one of the most comprehensive books ([6]) one can read the

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following: *"Although knot locations also affect shape, we know of no geometrically intuitive or mathematically simple interpretation of this effect ... "*. The aim of this paper is to present the geometrical and mathematical representation of the effects of knot modification for B-spline curves. The basic definitions are the following:

Definition 1 *The recursive function N_j^k given by the equations*

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}), \\ 0 & \text{otherwise} \end{cases}$$

$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u)$$

is called normalized B-spline basis function of order k (degree $k-1$). The numbers $u_j \leq u_{j+1} \in \mathbb{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

Definition 2 *The curve $\mathbf{s}(u)$ defined by*

$$\mathbf{s}(u) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u), \quad u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order k (degree $k-1$), where $N_l^k(u)$ is the l^{th} normalized B-spline basis function, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. The points $\mathbf{d}_i$ are called control points or DE BOOR*-points, while the polygon formed by these points is called control polygon. The arcs of this B-spline curve are called spans and the j^{th} span can be written as*

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \quad u \in [u_j, u_{j+1}).$$

Throughout this paper the following **properties** of the normalized B-spline basis functions will be used:

1. $N_j^k(u)$ is equal to 0 everywhere except on the interval $[u_j, u_{j+k})$.
2. At the r^{th} step of the recursive evaluation of $N_j^k(u)$ the following functions can occur:

$$N_{j+n}^{k-r}(u), \; r = 0, \ldots, k-1; \; n = 0, \ldots, r.$$

3. $\dot{N}_j^k(u) = (k-1)\left(\frac{1}{u_{j+k-1} - u_j} N_j^{k-1}(u) - \frac{1}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u)\right)$.
4. The modification of the knot u_i affects only the functions $N_{i-k}^k(u), \ldots, N_i^k(u)$, hence only the shape of the spans $\mathbf{s}_{i-k+1}(u), \ldots, \mathbf{s}_i(u), \ldots, \mathbf{s}_{i+k-2}(u)$ of the curve will be changed.

2. Effects of the modification of a knot value

When modifying the knot u_i, the basis functions and spans described in Property 4 will depend not only on u but on u_i as well. To emphasize this fact, they will be denoted by $N_i^k(u, u_i)$ and $\mathbf{s}_i(u, u_i)$. Fixing the second one of the two variables (i.e., the knot value $u_i = \widetilde{u}_i$) one can receive the original basis functions $N_i^k(u, \widetilde{u}_i) = N_i^k(u)$ and spans $\mathbf{s}_i(u, \widetilde{u}_i) = \mathbf{s}_i(u)$, but fixing the first variable (i.e., the parameter $u = \widetilde{u}$) the functions $N_i^k(\widetilde{u}, u_i)$ will not remain the standard basis functions any more, but some rational functions of u_i, while $\mathbf{s}_i(\widetilde{u}, u_i)$ can be interpreted as a curve on which a point of the original B-spline moves. More precisely, when modifying the knot u_i the point of the span $\mathbf{s}_j(u)$ associated with the fixed parameter value $\widetilde{u} \in [u_j, u_{j+1})$ will move along the curve

$$\mathbf{s}_j(\widetilde{u}, u_i) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(\widetilde{u}, u_i), \quad u_i \in [u_{i-1}, u_{i+1}].$$

Hereafter, we refer to this curve as the path of the point $\mathbf{s}_j(\widetilde{u})$. At the first part of this paper these functions and paths will be examined especially in terms of their degree.

Lemma 1 $N_{i-k}^k(\widetilde{u}, u_i)$, $\widetilde{u} \in [u_{i-m}, u_{i-m+1})$, $(m = 1, \ldots, k-1)$, $u_i \in [u_{i-1}, u_{i+1}]$ *is a rational function of degree* $k - m$ *in* u_i.

Proof. In the recursive Definition 1

$$N_{i-k}^k(\widetilde{u}, u_i) = \frac{\widetilde{u} - u_{i-k}}{u_{i-1} - u_{i-k}} N_{i-k}^{k-1}(\widetilde{u}, u_i) + \frac{u_i - \widetilde{u}}{u_i - u_{i-k+1}} N_{i-k+1}^{k-1}(\widetilde{u}, u_i)$$

the first term is independent of u_i because of Property 4, hence only the second term has to be considered. This fact is also valid for the further steps of the recursion:

$$\begin{aligned} &\vdots \\ N_{i-m-1}^{m+1}(\widetilde{u}, u_i) &= \frac{\widetilde{u} - u_{i-m-1}}{u_{i-1} - u_{i-m-1}} N_{i-m-1}^{m}(\widetilde{u}, u_i) + \frac{u_i - \widetilde{u}}{u_i - u_{i-m}} N_{i-m}^{m}(\widetilde{u}, u_i) \\ N_{i-m}^{m}(\widetilde{u}, u_i) &= \frac{\widetilde{u} - u_{i-m}}{u_{i-1} - u_{i-m}} N_{i-m}^{m-1}(\widetilde{u}, u_i) + \frac{u_i - \widetilde{u}}{u_i - u_{i-m+1}} N_{i-m+1}^{m-1}(\widetilde{u}, u_i). \end{aligned}$$

The first term of the right hand side of both equations above is constant because of Property 4. The second term of the last equation is equal to 0 due to Property 1. Thus u_i appears only in $k - m$ terms, at degree 1 everywhere, consequently, the degree of the function $N_{i-k}^k(\widetilde{u}, u_i)$ in u_i is $k - m$. □

Theorem 1 *The path*

$$\mathbf{s}_{i-m}(\widetilde{u}, u_i) = \sum_{l=i-m-k+1}^{i-m} N_l^k(\widetilde{u}, u_i)\mathbf{d}_l, \quad u_i \in [u_{i-1}, u_{i+1}]$$

is a rational curve of degree $k - m$ *with respect to* u_i, $\forall \widetilde{u} \in [u_{i-m}, u_{i-m+1})$, $(m = 1, \ldots, k-1)$.

Proof: The lower limit of the summation can be increased to $i - k$, since u_i has no effect on $N_l^k(\widetilde{u}, u_i)$ for $l < i - k$ (see Property 4). Hence only the functions

$$N_{i-k+z}^k(\widetilde{u}, u_i), \ z = 0, \ldots, k - m \tag{1}$$

have to be considered. The function $N_{i-k}^k(\widetilde{u}, u_i)$ is of degree $k - m$ because of Lemma 1. Thus it is sufficient to prove that the degree of the functions (1) is at most $k - m$, for $z > 0$.

At the r^{th} step of the recursion those functions which have influence on the functions mentioned above can be described in the following form (see Property 2):

$$\begin{aligned} N_{i-k+z+n}^{k-r}(\widetilde{u}, u_i) = &\frac{\widetilde{u} - u_{i-k+z+n}}{u_{i+z+n-r-1} - u_{i-k+z+n}} N_{i-k+z+n}^{k-r-1}(\widetilde{u}, u_i) + \\ &+ \frac{u_{i+z+n-r} - \widetilde{u}}{u_{i+z+n-r} - u_{i-k+z+n+1}} N_{i-k+z+n+1}^{k-r-1}(\widetilde{u}, u_i) \end{aligned}$$

for $r = 0, \ldots, k - 1$ and $n = 0, \ldots, r$. In this form u_i can occur in the following cases:

1. $i-k+z+n=i$, i.e., $z+n-k=0$, that is, the function $N_i^{k-r-1}(\widetilde{u},u_i)$ appears in the first term, but this function is equal to 0 on the interval $[u_{i-m},u_{i-m+1})$ for all permissible values of m (see Property 1).
2. $i+z+n-r-1=i$, that is $z+n=r+1$, hence the normalized B-spline basis function in the first term is $N_{i-(k-r-1)}^{k-r-1}(\widetilde{u},u_i)$. According to Lemma 1, the degree of this function in u_i is $k-m-r-1$, hence the degree of the first term can at most be $k-m-r\leq k-m$.
3. $i+z+n-r=i$, that is $z+n=r$, which corresponds to Case 2.
4. $i-k+z+n+1=i$, which corresponds to Case 1. □

Corollary 1 *For $m=k-1$, the resulted path is of degree 1, that is, if u_i runs from u_{i-1} to u_{i+1}, then the points of the span $\mathbf{s}_{i-k+1}(\widetilde{u},u_i)$ move along straight lines parallel to the side $\mathbf{d}_{i-k},\mathbf{d}_{i-k+1}$ of the control polygon.*

In this case the path has the following simple form:

$$\mathbf{s}_{i-k+1}\left(\widetilde{u},u_i\right)=\sum_{l=i-2(k-1)}^{i-k-1} N_l^k(\widetilde{u})\mathbf{d}_l+N_{i-k}^k(\widetilde{u},u_i)\mathbf{d}_{i-k}+N_{i-k+1}^k(\widetilde{u},u_i)\mathbf{d}_{i-k+1}\,,$$

where only the last two terms depend on u_i, and the normalized B-spline basis functions appearing in these terms can be written as

$$\begin{aligned} N_{i-k}^k(\widetilde{u},u_i) &= C_1(\widetilde{u})+C_2(\widetilde{u})\,\frac{u_i-\widetilde{u}}{u_i-u_{i-k+1}} \\ N_{i-k+1}^k(\widetilde{u},u_i) &= C_2(\widetilde{u})\,\frac{\widetilde{u}-u_{i-k+1}}{u_i-u_{i-k+1}} \end{aligned}$$

where

$$C_1(\widetilde{u})=\frac{\widetilde{u}-u_{i-k}}{u_{i-1}-u_{i-k}}N_{i-k}^{k-1}(\widetilde{u},u_i),\quad C_2(\widetilde{u})=\frac{\widetilde{u}-u_{i-k+1}}{u_{i-1}-u_{i-k+1}}N_{i-k+1}^{k-2}(\widetilde{u},u_i)$$

are constants not depending on u_i. Thus the u_i dependent part is

$$\begin{aligned} &\left(C_1(\widetilde{u})+C_2(\widetilde{u})\left(1-\frac{\widetilde{u}-u_{i-k+1}}{u_i-u_{i-k+1}}\right)\right)\mathbf{d}_{i-k}+C_2(\widetilde{u})\,\frac{\widetilde{u}-u_{i-k+1}}{u_i-u_{i-k+1}}\,\mathbf{d}_{i-k+1}= \\ &\quad=\left(C_1(\widetilde{u})+C_2(\widetilde{u})\right)\mathbf{d}_{i-k}+C_2(\widetilde{u})\,\frac{\widetilde{u}-u_{i-k+1}}{u_i-u_{i-k+1}}\,\left(\mathbf{d}_{i-k+1}-\mathbf{d}_{i-k}\right). \end{aligned}$$

As we have seen, the points of the span $\mathbf{s}_{i-k+1}(u,u_i)$ move along parallel straight lines. This change, however, cannot be described by an axial affine transformation, since the existence of the direction of an affinity would imply the existence of an axis containing fixed points which is not the case in general.

Until now the movement of those points and parts of the B-spline curve was clarified, the parameters of which are smaller than the knot value u_i subject to change. Similar statements hold for those parts of the B-spline curve, which correspond to the parameter values succeeding the knot u_i.

Lemma 2 *The function $N_i^k(\widetilde{u},u_i)$, $\widetilde{u}\in[u_{i+m},u_{i+m+1})$, $(m=0,\ldots,k-2)$, $u_i\in[u_{i-1},u_{i+1}]$ is a rational function of degree $k-m-1$ in u_i.*

Proof. The proof of this lemma is analogous to that of Lemma 1. □

Theorem 2 *The path*

$$\mathbf{s}_{i+m}(\widetilde{u}, u_i) = \sum_{l=i+m-k+1}^{i+m} N_l^k(\widetilde{u}, u_i)\mathbf{d}_l, \quad u_i \in [u_{i-1}, u_{i+1}]$$

is a rational curve of degree $k - m - 1$ with respect to u_i for all $\widetilde{u} \in [u_{i+m}, u_{i+m+1})$, $(m = 0, \ldots, k-2)$.

Proof. The upper limit of the summation can be decreased to i, since u_i has no influence on $N_l^k(\widetilde{u}, u_i)$ for $l > i$ (see Property 4). By using Lemma 2, the further part of the proof is analogous to that of Theorem 1. □

Corollary 2 *For $m = k - 2$, the path is of degree 1, that is, if u_i runs from u_{i-1} to u_{i+1} then the points of the span $\mathbf{s}_{i+k-2}(\widetilde{u}, u_i)$ move along straight lines parallel to the side $\mathbf{d}_{i-1}, \mathbf{d}_i$ of the control polygon.*

Now we can summarize our results based on Theorem 1 and 2 and their corollaries. Modifying the knot value u_i the points of the spans of a k^{th} order B-spline curve move along rational curves, the degree of which decreases symmetrically from $k - 1$ to 1 as the indices of the spans getting farther from i. Hence the points of the spans $\mathbf{s}_{i-k+1}(\widetilde{u}, u_i)$ and $\mathbf{s}_{i+k-2}(\widetilde{u}, u_i)$ move along straight lines parallel to the corresponding sides of the control polygon. Other parts of the curve remain unchanged (see Figure 1).

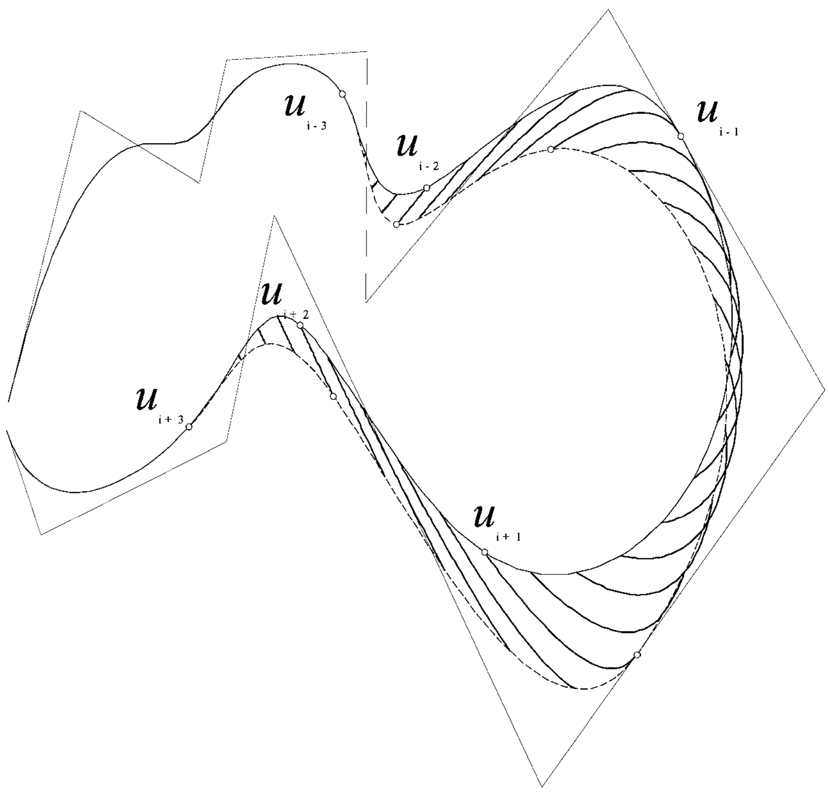

Figure 1: Paths of the points of a cubic B-spline curve when u_i runs from u_{i-1} to u_{i+1} and the two extreme positions of the curve: $u_i = u_{i-1}$ (solid line) and $u_i = u_{i+1}$ (dashed line)

3. The envelope of the family of B-spline curves

The modification of a knot value u_i results in a one-parameter family of B-spline curves

$$\mathbf{s}(u,u_i)=\sum_{l=0}^{n}\mathbf{d}_l N_l^k(u,u_i),\quad u\in[u_{k-1},u_{n+1}],\quad u_i\in[u_{i-1},u_{i+1}).$$

In case of $k=3$ the spans of the curves are parabolic arcs. It is well-known that the tangent lines of these arcs at the knot values coincide with the sides of the control polygon. Modifying a knot value u_i the tangent line remains the same, which can be interpreted as the side of the control polygon is the envelope of the family of these quadratic B-spline curves. In this section of the paper the generalization of this property will be proved for arbitrary k.

Theorem 3 *The family of the k^{th} order B-spline curves*

$$\mathbf{s}(u,u_i)=\sum_{l=0}^{n}\mathbf{d}_l N_l^k(u,u_i),\quad u\in[u_{k-1},u_{n+1}],\quad u_i\in[u_{i-1},u_{i+1}),\quad k>2,$$

has an envelope. This envelope is a B-spline curve of order $(k-1)$ and can be written in the form

$$\mathbf{b}(v)=\sum_{l=i-k+1}^{i-1}\mathbf{d}_l N_l^{k-1}(v),\quad v\in[v_{i-1},v_i],\ \textit{where}\ v_j=\begin{cases}u_j & \textit{if}\ j<i\\ u_{j+1} & \textit{if}\ j\geq i,\end{cases}$$

that is, the i^{th} knot value is removed from the knot vector (u_j) of the original curves.

Proof: We prove that the curve $\mathbf{b}(v)$ at its point corresponding to $v=u_i$ touches the curve $\mathbf{s}(u,u_i)$ at the point associated with $u=u_i$, i.e., they have a point and a tangent line on common.

Based on Definition 1 the span

$$\mathbf{s}_i(u,u_i)=\sum_{l=i-k+1}^{i}\mathbf{d}_l N_l^k(u,u_i),\ u\in[u_i,u_{i+1}] \tag{2}$$

the starting point of which is $\mathbf{s}(u_i,u_i)$ can be written in the form

$$\mathbf{s}_i(u,u_i)=\sum_{l=i-k+1}^{i}\mathbf{d}_l\left(\frac{u-u_l}{u_{l+k-1}-u_l}N_l^{k-1}(u)+\frac{u_{l+k}-u}{u_{l+k}-u_{l+1}}N_{l+1}^{k-1}(u)\right). \tag{3}$$

At the specific parameter u_i the value of this function is

$$\mathbf{s}_i(u_i,u_i)=\sum_{l=i-k+1}^{i-1}\mathbf{d}_l\left(\frac{u_i-u_l}{u_{l+k-1}-u_l}N_l^{k-1}(u_i)+\frac{u_{l+k}-u_i}{u_{l+k}-u_{l+1}}N_{l+1}^{k-1}(u_i)\right) \tag{4}$$

(the upper limit of the summation can be decreased to $i-1$, since $N_i^{k-1}(u_i)=N_{i+1}^{k-1}(u_i)=0$).

Now we insert the knot value u_i between v_{i-1} and v_i ($v_{i-1}=u_{i-1}\leq u_i\leq u_{i+1}=v_i$) by the BÖHM's insertion algorithm (cf. [2]). The new knot vector is

$$\hat{v}_j=\begin{cases}v_j=u_j & \text{if}\ j<i\\ u_i & \text{if}\ j=i\\ v_{j-1}=u_j & \text{if}\ j>i\end{cases} \tag{5}$$

For the normalized B-spline basis functions $N_l^{k-1}(v)$ and $\hat{N}_l^{k-1}(v)$, defined over the knot vectors (v_j) and $(\hat{v}_j)$ respectively, the following relation holds:

$$N_l^{k-1}(v) = \begin{cases} \hat{N}_l^{k-1}(v), & \text{if } l < i-k+1; \\ \dfrac{u_i - \hat{v}_l}{\hat{v}_{l+k-1} - \hat{v}_l}\hat{N}_l^{k-1}(v) + \dfrac{\hat{v}_{l+k} - u_i}{\hat{v}_{l+k} - \hat{v}_{l+1}}\hat{N}_{l+1}^{k-1}(v), & \text{if } l = i-k+1,\ldots,i-1; \\ \hat{N}_{l+1}^{k-1}(v), & \text{if } l > i-1. \end{cases}$$

Based on this fact the following form can be obtained

$$\mathbf{b}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l \left(\frac{u_i - \hat{v}_l}{\hat{v}_{l+k-1} - \hat{v}_l}\hat{N}_l^{k-1}(v) + \frac{\hat{v}_{l+k} - u_i}{\hat{v}_{l+k} - \hat{v}_{l+1}}\hat{N}_{l+1}^{k-1}(v) \right), \quad v \in [\hat{v}_{i-1}, \hat{v}_{i+1}),$$

and using (5) this can be written in the form (since $\hat{v}_j = u_j \ \forall j$)

$$\mathbf{b}(u) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l \left(\frac{u_i - u_l}{u_{l+k-1} - u_l}N_l^{k-1}(u) + \frac{u_{l+k} - u_i}{u_{l+k} - u_{l+1}}N_{l+1}^{k-1}(u) \right) \quad u \in [u_{i-1}, u_{i+1}). \tag{6}$$

Comparing (4) and (6) one can see that $\mathbf{b}(u_i) = \mathbf{s}_i(u_i, u_i)$ holds.

The derivative of the curve (6) with respect to u is

$$\dot{\mathbf{b}}(u) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l \left(\frac{u_i - u_l}{u_{l+k-1} - u_l}\dot{N}_l^{k-1}(u) + \frac{u_{l+k} - u_i}{u_{l+k} - u_{l+1}}\dot{N}_{l+1}^{k-1}(u) \right). \tag{7}$$

Based on (2), the derivative of the curve $\mathbf{s}_i(u, u_i)$ with respect to u is

$$\dot{\mathbf{s}}_i(u, u_i) = \sum_{l=i-k+1}^{i} \mathbf{d}_l \dot{N}_l^k(u, u_i), \tag{8}$$

while on the other hand, using (3)

$$\begin{aligned} \dot{\mathbf{s}}_i(u, u_i) \ = \ & \sum_{l=i-k+1}^{i} \mathbf{d}_l \left(\frac{1}{u_{l+k-1} - u_l}N_l^{k-1}(u) - \frac{1}{u_{l+k} - u_{l+1}}N_{l+1}^{k-1}(u) + \right. \\ & \left. + \frac{u - u_l}{u_{l+k-1} - u_l}\dot{N}_l^{k-1}(u) + \frac{u_{l+k} - u}{u_{l+k} - u_{l+1}}\dot{N}_{l+1}^{k-1}(u) \right) \end{aligned}$$

holds. Applying Property 3, this can be written as

$$\dot{\mathbf{s}}_i(u, u_i) = \sum_{l=i-k+1}^{i} \mathbf{d}_l \left(\frac{u - u_l}{u_{l+k-1} - u_l}\dot{N}_l^{k-1}(u) + \frac{u_{l+k} - u}{u_{l+k} - u_{l+1}}\dot{N}_{l+1}^{k-1}(u) + \frac{1}{k-1}\dot{N}_l^k(u) \right). \tag{9}$$

Based on (8) and (9) one can write

$$\frac{u - u_l}{u_{l+k-1} - u_l}\,\dot{N}_l^{k-1}(u) + \frac{u_{l+k} - u}{u_{l+k} - u_{l+1}}\,\dot{N}_{l+1}^{k-1}(u) = \frac{k-2}{k-1}\,\dot{N}_l^k(u).$$

Hence from (8) and (7) we obtain

$$\dot{\mathbf{b}}(u_i) = \frac{k-2}{k-1}\,\dot{\mathbf{s}}_i(u_i, u_i). \qquad \square$$

The envelope is illustrated in Figure 2.

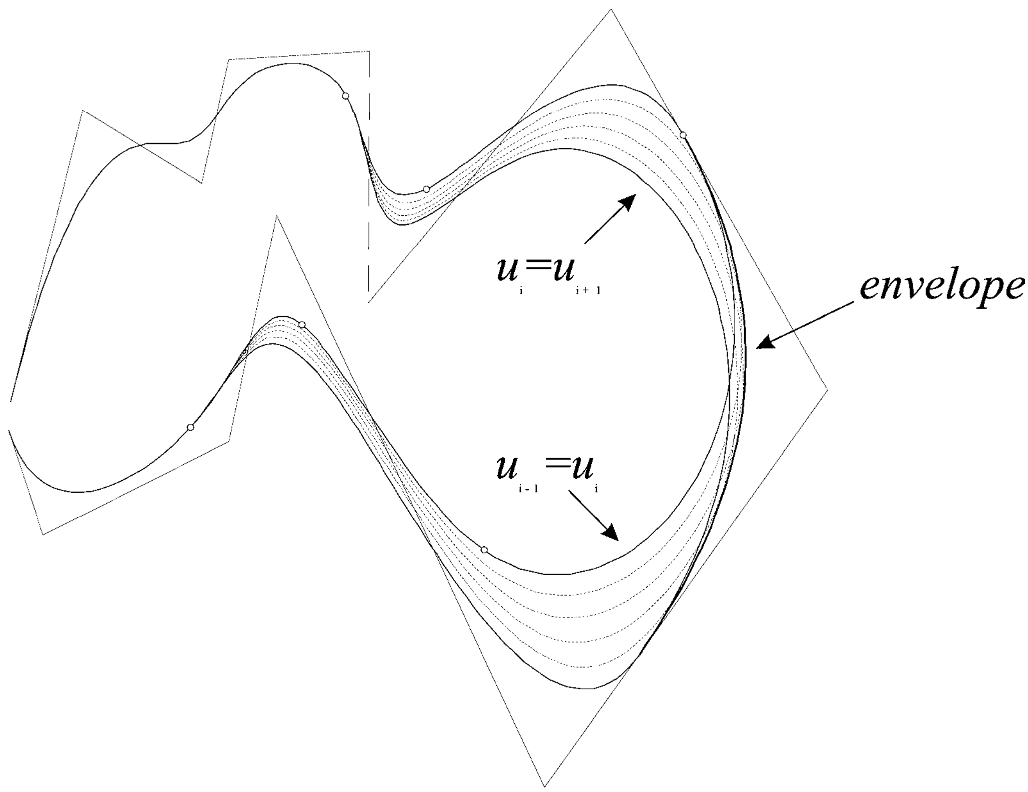

Figure 2: Envelope of the family of cubic B-spline curves when u_i runs from u_{i-1} to u_{i+1}

4. Conclusions and further research

Some geometrical properties of knot modification have been presented in this paper. Since we restricted our consideration only to the basic type of B-spline curves, a natural extension of these results could be the examination of rational B-spline curves, for which probably similar results will be achieved. Modification of knots of higher multiplicity or more than one knot at a time are also of interest. The most essential direction of our further research, however, must be to find easy-to-use tools for knot-based shape control of B-spline and NURBS curves.

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