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On parametrization of interpolating curves

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Abstract

In parametric curve interpolation there is given a sequence of data points and corresponding parameter values (nodes), and we want to find a parametric curve that passes through data points at the associated parameter values. We consider those interpolating curves that are described by the combination of control points and blending functions. We study paths of control points and points of the interpolating curve obtained by the alteration of one node. We show geometric properties of quadratic Bézier interpolating curves with uniform and centripetal parameterizations. Finally, we propose geometric methods for the interactive modification and specification of nodes for interpolating Bézier curves.
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1. Introduction

Interpolation of a sequence of points is one of the most basic tools in computer aided geometric design and its applications. The exact problem statement is as follows. We are given a sequence of points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ and corresponding parameter values $u_0, u_1, \ldots, u_n$, and we want to find those curves $\mathbf{g}(u)$, $u \in [u_0, u_n]$ for which $\mathbf{g}(u_i) = \mathbf{p}_i$, $(i = 0, 1, \ldots, n)$. Values u_i are also called nodes. The solution of this problem is quite straightforward and can be found in all comprehensive books of the field, e.g., in [2,5,14]. But even in the simplest cases there is a point in the algorithm which is critical for the final result: the parameterization of the interpolating curve, i.e., the choice of nodes u_i $(i = 0, \ldots, n)$. The problem is that while the specification of data points $\mathbf{p}_i$ is intuitive, users of CAD systems can do it easily, the specification of parameter values u_i is not intuitive at all, although they have a significant influence on the shape of the resulted curve. They can be specified in infinite ways and there is no universally optimal solution.

A vast number of papers deal with the possible strategies, from the simplest equidistant parameterization through chord length and centripetal (cf. [10]) methods to some sophisticated techniques as affine invariant parameterization (cf. [4]). A special problem of closed curves is discussed in [11], while a more general approach of shape-preserving interpolation can be found, e.g., in [9] and references therein. There are recent results declaring the “best”

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parameterization [8], however, the perfect choice of parameterization still seems to be problem- and user-dependent. Hence it is essential to be aware of the effects of parameterization on the shape of the curve.

While all the above-mentioned papers deal with the problem from an analytical, numerical point of view, hardly anything we know about the geometric aspects of parameterization, especially about the geometric effects of the alteration of one or more nodes. One of the aims of this paper is to describe the geometric aspects of node alteration, independently of the actual parameterization technique.

On the other hand, while control point repositioning is the most frequent way of interactive modification in curve approximation, there is no similar way for geometrically intuitive shape control of interpolation curves. While interactive techniques have also some marks in the literature (cf. [12,13]), there is no such an intuitive way for curve interpolation as for approximation. Based on our theoretical results we propose control point-based techniques for the shape control of interpolating Bézier curves.

The paper is organized as follows. Section 2 describes general results where there are no assumptions for the blending functions of the interpolation curve. In Section 3 curves with polynomial blending functions are discussed, while in Sections 4 and 5 consequences for quadratic and cubic Bézier interpolation are considered, respectively. Finally, in Section 6 we provide an interactive way of control point-based shape modification of interpolating Bézier curves.

2. The general case

We are given data points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ and corresponding parameter values $u_0 < u_1 < \cdots < u_n$. We want to find an interpolating curve of the form

$$\mathbf{d}(u) = \sum_{j=0}^{n} F_j(u)\mathbf{d}_j, \quad u \in [u_0, u_n], \tag{1}$$

i.e., conditions

$$\mathbf{d}(u_i) = \mathbf{p}_i \quad (i = 0, 1, \ldots, n)$$

have to be fulfilled. $\mathbf{d}_j$ are called control points and $F_j(u)$ are blending functions of the interpolation method. In order to calculate with numbers of small absolute value, that decreases the error due to rounding, parameter values are normalized, i.e., $u_0 = 0 < u_1 < \cdots < u_n = 1$.

On the grounds of these assumptions we obtain the system of equations

$$\begin{bmatrix} F_0(u_0) & F_1(u_0) & \cdots & F_n(u_0) \\ F_0(u_1) & F_1(u_1) & \cdots & F_n(u_1) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(u_n) & F_1(u_n) & \cdots & F_n(u_n) \end{bmatrix} \begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_n \end{bmatrix} \tag{2}$$

for the unknown control points $\mathbf{d}_j$. It has a unique solution if the determinant D of the system's matrix A does not vanish. This is the only condition for the functions $F_j(u)$. In this case the system can be solved by means of Cramer's rule and we obtain control points

$$\mathbf{d}_j = \frac{\sum_{i=0}^{n} A_{ij}\mathbf{p}_i}{D} \quad (j = 0, 1, \ldots, n), \tag{3}$$

where A_{ij} is the signed minor of element a_{ij} in A. (Certainly, in practice system (2) is solved by other methods but for our further study this form is convenient.)

Nodes u_i of the domain have a global effect on the shape of the interpolating curve (1), i.e., regardless of the interpolation method, if one of them is altered the shape of the whole curve will be changed.

If we let the node u_i vary in Eq. (1), we obtain the path

$$\mathbf{g}(u, u_i), \quad u_i \in (u_{i-1}, u_{i+1}) \tag{4}$$

of the point that corresponds to the parameter u. Obviously, this path passes through the point $\mathbf{p}_i$ if $u \in (u_{i-1}, u_{i+1})$.

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Analogously, when varying the value u_i in Eq. (3) all the control points $\mathbf{d}_j$ $(j = 0, 1, \ldots, n)$ vary and we obtain a curve

$$\mathbf{d}_j(u_i), \quad u_i \in (u_{i-1}, u_{i+1})$$

for each control point that we call the path of the control point $\mathbf{d}_j$ with respect to u_i.

2.1. Fixed points

We let node u_i vary between u_{i-1} and u_{i+1} while the rest is held fixed, and we want to find such a point on the joining line of two control points $\mathbf{d}_j$ and $\mathbf{d}_{j+z}$ $(0 \leqslant j < n, 0 < z \leqslant n - j)$ that is independent of the specification of u_i.

We remove row $i + 1$ (that contains $F_j(u_i)$) from the system of equations (2), thus we obtain a system that has one more unknowns than equations. This under-determined system has a unique solution for the combinations of unknowns $\mathbf{d}_j$ and $\mathbf{d}_{j+z}$ of the form

$$\delta_j(U_{-i})\mathbf{d}_j + \delta_{j+z}(U_{-i})\mathbf{d}_{j+z} = \sum_{l=0, l \neq i}^{n} \alpha_l(U_{-i})\mathbf{p}_l,$$

where $U_{-i} = \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n-1}\}$ and δ_j are functions of U_{-i}. Dividing both sides by the non-zero sum $\delta_j(U_{-i}) + \delta_{j+z}(U_{-i})$ and using the notation

$$\lambda_j(U_{-i}) = \frac{\delta_j(U_{-i})}{\delta_j(U_{-i}) + \delta_{j+z}(U_{-i})}$$

we obtain

$$\lambda_j(U_{-i})\mathbf{d}_j + (1 - \lambda_j(U_{-i}))\mathbf{d}_{j+z} = \frac{\sum_{l=0, l \neq i}^{n} \alpha_l(U_{-i})\mathbf{p}_l}{\delta_j(U_{-i}) + \delta_{j+z}(U_{-i})} \tag{5}$$

(α_l are functions of U_{-i}). Thus, we get a point on the joining line of $\mathbf{d}_j$ and $\mathbf{d}_{j+z}$ that is invariant under the change of u_i.

Applying considerations above, we can find fixed points as barycentric combination of more than two control points, if we let more than one node vary. The number of varying nodes can be as much as n.

When u_i is altered the shape of the interpolating curve continuously changes and points of the curve move along paths (4). The following theorem is on the relation between interpolating curves that correspond to two arbitrarily fixed values of u_i.

Theorem 1. *Any two interpolating curves that correspond to two arbitrarily fixed values of u_i can be obtained by a functional translation from each other.*

Proof. It is known that if a control point $\mathbf{d}_l$ of a curve of type (1) is translated by the vector $\mathbf{t}$, the transformed curve is of the form

$$\mathbf{d}(u) = \sum_{j=0}^{n} F_j(u)\mathbf{d}_j + F_l(u)\mathbf{t}, \tag{6}$$

i.e., the translation of a control point implies a functional translation of the curve (a translation where the shift vector varies by the function $F_l(u)$).

Fixed point (5) on the joining line of $\mathbf{d}_l$ and $\mathbf{d}_{l+z}$ is a barycentric combination of the connected two points and is independent of u_i, therefore lines joining any two positions $\mathbf{d}_l(u_i)$ (a point on the path of $\mathbf{d}_l$ with respect to u_i) and $\mathbf{d}_l(\bar{u}_i)$ as well as lines joining $\mathbf{d}_{l+z}(u_i)$ and $\mathbf{d}_{l+z}(\bar{u}_i)$ are parallel. Let $\mathbf{t}$ be this common direction. Consequently, the joining line of the two positions, that correspond to u_i and $\bar{u}_i$, of any control point is parallel to $\mathbf{t}$. Thus, corresponding points of the two interpolating curves are on lines that are parallel to $\mathbf{t}$, due to Eq. (6). The concerned control points $\mathbf{d}_j$

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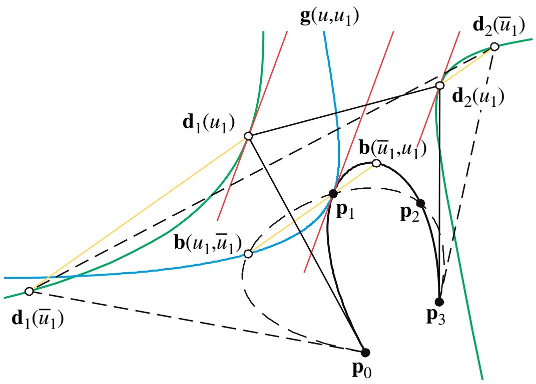

Fig. 1. Tangent lines of paths and of the interpolating curve are parallel.

are translated by the vector $\mu_j \mathbf{t}$ and the modified curve is

$$\sum_{j=0}^{n} F_j(u)\mathbf{d}_j + \left(\sum_{j=0}^{n} \mu_j F_j(u)\right)\mathbf{t}$$

that completes the proof. □

Theorem 2. *When the node u_i is altered the tangent line of the interpolating curve at $\mathbf{d}(u_i) = \mathbf{p}_i$ coincides with the tangent line of the path $\mathbf{g}(u_i, u_i)$ at $\mathbf{p}_i$, and this common tangent line is parallel to the tangent lines of paths at u_i of all concerned control points.*

Proof. Let us consider two values u_i and $\bar{u}_i$ of the varying node. In Theorem 1 we have shown that chords joining the points $\mathbf{d}_l(u_i)$ and $\mathbf{d}_l(\bar{u}_i)$ on the path of $\mathbf{d}_l$ are parallel $\forall l$. We denote this direction by $\mathbf{t}$, and the family of interpolating curves obtained by the modification of u_i by $\mathbf{b}(u, u_i)$. Members $\mathbf{b}(u, u_i)$ and $\mathbf{b}(u, \bar{u}_i)$ of the family of the interpolating curves are functional translations of each other, parallel to $\mathbf{t}$. During this translation the correspondence is

$$\mathbf{p}_i = \mathbf{b}(u_i, u_i) \to \mathbf{b}(u_i, \bar{u}_i),$$

$$\mathbf{b}(\bar{u}_i, u_i) \to \mathbf{b}(\bar{u}_i, \bar{u}_i) = \mathbf{p}_i$$

and points $\mathbf{b}(\bar{u}_i, u_i)$, $\mathbf{b}(u_i, u_i)$ and $\mathbf{b}(u_i, \bar{u}_i)$ are colinear, and their line is parallel to $\mathbf{t}$. (Fig. 1 illustrates this for cubic Bézier interpolation, $i = 1$.) Taking the limit $\bar{u}_i \to u_i$ the proof is completed. □

2.2. *Blending functions sum to 1*

Theorem 3. *If*

$$\sum_{j=0}^{n} F_j(u) = 1 \quad (j = 0, 1, \ldots, n), \tag{7}$$

then control points $\mathbf{d}_j$ and the interpolating curve itself are barycentric combinations of data points $\mathbf{p}_i$.

Proof. Because of equality (3) control points $\mathbf{d}_j$ are of the form

$$\mathbf{d}_j = \sum_{i=0}^{n} \alpha_{ij}\mathbf{p}_i, \quad \alpha_{ij} = \frac{A_{ij}}{D}, \quad (j = 0, 1, \ldots, n).$$

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First we show that

$$\sum_{i=0}^{n} A_{ij} = D \quad (j = 0, 1, \ldots, n) \tag{8}$$

if assumption (7) holds. Let us add to the jth column of matrix A all the rest of its columns, whose process does not change the determinant. Due to our assumption, all elements of the jth column become 1. Expanding the determinant along its jth column we obtain equality (8), as a result of which

$$\sum_{i=0}^{n} \alpha_{ij} = 1. \tag{9}$$

The interpolating curve can be written in the form

$$\mathbf{d}(u) = \sum_{j=0}^{n} F_j(u)\mathbf{d}_j = \sum_{j=0}^{n} F_j(u) \sum_{i=0}^{n} \alpha_{ij}\mathbf{p}_i,$$

and the sum of the coefficients of data points $\mathbf{p}_i$ is

$$\sum_{j=0}^{n} F_j(u) \sum_{i=0}^{n} \alpha_{ij} = 1$$

due to equalities (9) and (7). □

Definition 4. The i discriminant of curve (1) is the curve

$$\mathbf{c}_i(u) = -\frac{\sum_{j=0, j\neq i}^{n} \dot{F}_j(u)\mathbf{d}_j}{\dot{F}_i(u)},$$

where $\dot{F}_j(u)$ denotes the derivative of $F_j(u)$.

Discriminant curves are used to detect singularities of control point-based parametric curves, cf. [7].

Proposition 5. *If a curve is a barycentric combination of its control points then all of its discriminants are also barycentric combinations of them.*

Proof. If $\sum_{j=0}^{n} F_j(u) = 1 \Rightarrow \sum_{j=0}^{n} \dot{F}_j(u) = 0 \Rightarrow \dot{F}_i(u) = -\sum_{j=0, j\neq i}^{n} \dot{F}_j(u)$ that completes the proof. □

3. Polynomial blending functions

If functions $F_j(u)$ form a basis of degree n polynomial space, equation system (2) has a unique solution if nodes u_i are different. Node u_k, $(k = 0, 1, \ldots, n)$ appears only in the row $k + 1$ of A, therefore polynomials of u_k are not multiplied when expanding the determinant. Thus, any $\mathbf{d}_j$ is a rational function of degree at most n in u_k. The same holds for the paths of points of the interpolating curve, since they are linear combinations of paths of control points.

Paths (both of control points and of points of the interpolating curve) share the same points at infinity. These paths have a point at infinity where the determinant D vanishes. When modifying the node u_k, $D = 0$ occurs if $u_k = u_{k-1}$ or u_{k+1}, i.e., when two rows of A are identical. Consequently, there are only two non-vanishing signed minors and these differ only in their sign. Therefore, path of control point $\mathbf{d}_i$ $(i = 1, 2, \ldots, n-1)$ has two points at infinity the directions of which are $\mathbf{p}_k - \mathbf{p}_{k-1}$ and $\mathbf{p}_k - \mathbf{p}_{k+1}$ (points at infinity can be represented by directions in the Euclidean space). The same holds for the paths of points of the interpolating curve.

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4. Quadratic Bézier interpolation

Given data points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ we consider the Bézier representation

$$\mathbf{b}(u)=\sum_{i=0}^{2}\mathbf{b}_i B_i^2(u), \quad u\in[0,1] \tag{10}$$

of the interpolating parabolic arc, where $B_i^2(u)$ denotes the ith quadratic Bernstein polynomial. As a special case of (2) we obtain control points of the interpolating quadratic Bézier curve of the form

$$\begin{aligned}\mathbf{b}_0&=\mathbf{p}_0,\\ \mathbf{b}_1&=\frac{\mathbf{p}_1-\mathbf{p}_0B_0^2(u_1)-\mathbf{p}_2B_2^2(u_1)}{B_1^2(u_1)},\\ \mathbf{b}_2&=\mathbf{p}_2\end{aligned} \tag{11}$$

by the assumptions $\mathbf{b}(u_i)=\mathbf{p}_i$ $(i=0,1,2)$ and $u_0=0, u_2=1$. Applying the identity $\sum_{i=0}^{2}B_i^2(u)\equiv 1$ the path of the control point $\mathbf{b}_1$ subject to the alteration of u_1 is

$$\mathbf{b}_1(u_1)=\frac{(\mathbf{p}_1-\mathbf{p}_0)B_0^2(u_1)+\mathbf{p}_1B_1^2(u_1)+(\mathbf{p}_1-\mathbf{p}_2)B_2^2(u_1)}{B_1^2(u_1)} \tag{12}$$

$$=\mathbf{p}_1+(\mathbf{p}_1-\mathbf{p}_0)\frac{1-u_1}{2u_1}+(\mathbf{p}_1-\mathbf{p}_2)\frac{u_1}{2(1-u_1)}. \tag{13}$$

This is a quadratic rational curve with two points at infinity ($u_1=0, u_1=1$), therefore it is a hyperbolic arc with center $\mathbf{p}_1$ and asymptotic directions $(\mathbf{p}_1-\mathbf{p}_0)$ and $(\mathbf{p}_1-\mathbf{p}_2)$.

Path of points of curve (10) are of the form

$$\begin{aligned}\mathbf{g}(u,u_1)&=\mathbf{p}_0B_0^2(u)+\mathbf{b}_1(u_1)B_1^2(u)+\mathbf{p}_2B_2^2(u)\\ &=\sum_{i=0}^{2}\mathbf{p}_iB_i^2(u)+\left((\mathbf{p}_1-\mathbf{p}_0)\frac{1-u_1}{2u_1}+(\mathbf{p}_1-\mathbf{p}_2)\frac{u_1}{2(1-u_1)}\right)B_1^2(u) \quad u_1\in(0,1).\end{aligned} \tag{14}$$

Thus, these paths can be obtained from the hyperbolic arc (13) by shrinking/enlargement and translation. Consequently, these paths are hyperbolic arcs the center of which is

$$\mathbf{p}(u)=\sum_{i=0}^{2}\mathbf{p}_iB_i^2(u)$$

and the asymptotic directions are that of the curve $\mathbf{b}_1(u_1)$, cf. Fig. 2. These paths pass through the point $\mathbf{p}_1$ at $u_1=u$ for all u.

The so-called exponential parameterization includes several well-known parameterization. Its general form is

$$u_0=0, \quad u_i=u_{i-1}+\frac{|\mathbf{p}_i-\mathbf{p}_{i-1}|^e}{\sum_{j=1}^{n}|\mathbf{p}_j-\mathbf{p}_{j-1}|^e} \quad (i=1,2,\ldots,n), \quad e\in[0,1]. \tag{15}$$

The $e=0$ case is the uniform parameterization, the $e=1$ case is the chord length parameterization while in case of $e=\frac{1}{2}$ we obtain the centripetal parameterization.

In the following two subsections we show some interesting geometric properties of quadratic Bézier interpolating curves with uniform and centripetal parameterization.

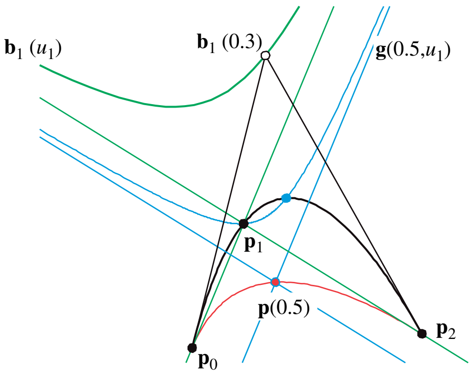

Fig. 2. Paths of points of the interpolating quadratic Bézier curve are hyperbolas the center of which are on the parabolic arc whose control points are the points to be interpolated ($u_1 = 0.3$, $u = 0.5$).

4.1. Uniform parameterization

Area constraints are of interest in practice, e.g., in cross sectional design, cf. [3,1]. Using Theorem 4 in [6] one can compute the area of the plane region bounded by a Bézier curve and the straight line segments joining its endpoints with the origin. By means of this, we obtain the following result for the uniform parametrization.

Theorem 6. *The signed area of the plane region bounded by the parabolic arc that interpolates data points* $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ *and the straight line segments joining its endpoints with the origin is minimal if the parameterization is uniform.*

Proof. The area of the parabolic arc determined by control points $\mathbf{p}_0, \mathbf{b}_1, \mathbf{p}_2$ is

$$T = \tfrac{1}{6}(2(\mathbf{p}_0 \wedge \mathbf{b}_1(u_1)) + (\mathbf{p}_0 \wedge \mathbf{p}_2) + 2(\mathbf{b}_1(u_1) \wedge \mathbf{p}_2)).$$

Considering Eq. (11) we obtain

$$T = \frac{(\mathbf{p}_0 \wedge \mathbf{p}_1) + (\mathbf{p}_1 \wedge \mathbf{p}_2) - (\mathbf{p}_0 \wedge \mathbf{p}_2)}{3B_1^2(u_1)} + \frac{(\mathbf{p}_0 \wedge \mathbf{p}_2)}{2}. \tag{16}$$

This area is minimal when $B_1^2(u_1)$ takes its maximum value, i.e., at $u_1 = 0.5$.

The minimal area is

$$T_{\min} = \frac{2}{3}((\mathbf{p}_0 \wedge \mathbf{p}_1) + (\mathbf{p}_1 \wedge \mathbf{p}_2) - (\mathbf{p}_0 \wedge \mathbf{p}_2)) + \frac{(\mathbf{p}_0 \wedge \mathbf{p}_2)}{2}. \qquad \square$$

The area of interpolating quadratic Bézier curves varies in the range $[|T_{\min}|, \infty)$. For an arbitrarily chosen area $T \in [|T_{\min}|, \infty)$ one can compute the corresponding u_1 by solving the quadratic equation (16). There will always be two solutions, since $B_1^2(1/2 - \delta) = B_1^2(1/2 + \delta)\ \forall \delta \in \mathbb{R}$.

4.2. Centripetal parameterization

Theorem 7. *In case of centripetal parameterization control point* $\mathbf{b}_1$ *is on the axis of the hyperbola* (13).

Proof. Let us consider the path (13) of control point $\mathbf{b}_1$ and introduce notations $l_1 = |\mathbf{p}_1 - \mathbf{p}_0|$, $l_2 = |\mathbf{p}_1 - \mathbf{p}_2|$. By means of this, in case of centripetal parameterization

$$u_1 = \frac{\sqrt{l_1}}{\sqrt{l_1} + \sqrt{l_2}},$$

that implies

$$\frac{1-u_1}{2u_1}=\frac{\sqrt{l_2}}{2\sqrt{l_1}},\qquad \frac{u_1}{2(1-u_1)}=\frac{\sqrt{l_1}}{2\sqrt{l_2}},$$

$$(\mathbf{p}_1-\mathbf{p}_0)\frac{1-u_1}{2u_1}=(\mathbf{p}_1-\mathbf{p}_0)\frac{\sqrt{l_2}}{2\sqrt{l_1}}=\frac{\sqrt{l_1l_2}}{2}\frac{(\mathbf{p}_1-\mathbf{p}_0)}{l_1},$$

$$(\mathbf{p}_1-\mathbf{p}_2)\frac{u_1}{2(1-u_1)}=(\mathbf{p}_1-\mathbf{p}_2)\frac{\sqrt{l_1}}{2\sqrt{l_2}}=\frac{\sqrt{l_1l_2}}{2}\frac{(\mathbf{p}_1-\mathbf{p}_2)}{l_2}.$$

Thus, point $\mathbf{b}_1(\sqrt{l_1}/(\sqrt{l_1}+\sqrt{l_2}))$ is on the angular bisector of asymptotes, i.e., located on the axis. □

5. Cubic Bézier interpolation

Given data points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ we consider the Bézier representation

$$\mathbf{b}(u)=\sum_{i=0}^{3}\mathbf{b}_iB_i^3(u),\qquad u\in[0,1] \tag{17}$$

of the interpolating cubic, where $B_i^3(u)$ denotes the ith cubic Bernstein polynomial. By the assumption $u_0=0$, $u_3=1$ the system of equations (2) is reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ B_0^3(u_1) & B_1^3(u_1) & B_2^3(u_1) & B_3^3(u_1)\\ B_0^3(u_2) & B_1^3(u_2) & B_2^3(u_2) & B_3^3(u_2)\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{d}_0\\ \mathbf{d}_1\\ \mathbf{d}_2\\ \mathbf{d}_3\end{bmatrix}=\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}. \tag{18}$$

5.1. Fixed points

Removing the second row from (18), for the sum of $\mathbf{d}_1$ and $\mathbf{d}_2$ we obtain the equality

$$B_1^3(u_2)\mathbf{d}_1+B_2^3(u_2)\mathbf{d}_2=\mathbf{p}_2-B_0^3(u_2)\mathbf{p}_0-B_3^3(u_2)\mathbf{p}_3.$$

From this

$$\lambda_1(u_2)\mathbf{d}_1+(1-\lambda_1(u_2))\mathbf{d}_2=\mathbf{p}_2+\frac{(1-u_2)^2}{3u_2}(\mathbf{p}_2-\mathbf{p}_0)+\frac{u_2^2}{3(1-u_2)}(\mathbf{p}_2-\mathbf{p}_3),$$

where

$$\lambda_1(u_2)=\frac{B_1^3(u_2)}{B_2^3(u_2)+B_1^3(u_2)}=1-u_2.$$

Thus, point $(1-u_2)\mathbf{d}_1+u_2\mathbf{d}_2$ remains unchanged when u_1 is altered. Analogously, we can compute other fixed points.

5.2. Asymptotes of paths of control points

The determinant of system (18) is

$$D=B_1^3(u_1)B_2^3(u_2)-B_2^3(u_1)B_1^3(u_2).$$

On the bases of Eqs. (18) control point $\mathbf{d}_1$ can be written in the form

$$\mathbf{d}_1(u_1,u_2)=\alpha_{01}\mathbf{p}_0+\alpha_{11}\mathbf{p}_1+\alpha_{21}\mathbf{p}_2+\alpha_{31}\mathbf{p}_3, \tag{19}$$

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where

$$\alpha_{01} = \frac{B_2^3(u_1)B_0^3(u_2) - B_0^3(u_1)B_2^3(u_2)}{D} = \frac{1}{3}\frac{2u_1u_2 - u_1 - u_2}{u_1u_2},$$

$$\alpha_{11} = \frac{B_2^3(u_2)}{D} = \frac{1}{3}\frac{u_2}{u_1(1-u_1)(u_2-u_1)},$$

$$\alpha_{21} = -\frac{B_2^3(u_1)}{D} = -\frac{1}{3}\frac{u_1}{u_2(1-u_2)(u_2-u_1)},$$

$$\alpha_{31} = \frac{B_2^3(u_1)B_3^3(u_2) - B_3^3(u_1)B_2^3(u_2)}{D} = \frac{1}{3}\frac{u_1u_2}{(1-u_1)(1-u_2)}.$$

In order to determine the asymptotes of the path with respect to u_1 of control point $\mathbf{d}_1$ we apply the equality $\alpha_{11} = 1 - \alpha_{01} - \alpha_{21} - \alpha_{31}$ (that is a consequence of Theorem 3) and take the limits $u_1 \to 0$ and $u_1 \to u_2$. The resulted limits are

$$\lim_{u_1 \to 0} \mathbf{d}_1(u_1, u_2) = \mathbf{p}_1 + (\mathbf{p}_0 - \mathbf{p}_1)_\infty, \tag{20}$$

$$\lim_{u_1 \to u_2} \mathbf{d}_1(u_1, u_2) = \mathbf{p}_1 + \frac{B_3^3(u_2)(\mathbf{p}_3 - \mathbf{p}_1) - 2B_0^3(u_2)(\mathbf{p}_0 - \mathbf{p}_1)}{B_1^3(u_2)} + (\mathbf{p}_2 - \mathbf{p}_1)_\infty. \tag{21}$$

To obtain the asymptotes of path with respect to u_2 of $\mathbf{d}_1$ we apply the identity $\alpha_{21} = 1 - \alpha_{01} - \alpha_{11} - \alpha_{31}$ and take the limits

$$\lim_{u_2 \to 1} \mathbf{d}_1(u_1, u_2) = \mathbf{p}_2 + \frac{(\mathbf{p}_1 - \mathbf{p}_2) - B_0^3(u_1)(\mathbf{p}_0 - \mathbf{p}_2)}{B_1^3(u_1)} + (\mathbf{p}_3 - \mathbf{p}_2)_\infty,$$

$$\lim_{u_2 \to u_1} \mathbf{d}_1(u_1, u_2) = \mathbf{p}_2 + \frac{B_3^3(u_1)(\mathbf{p}_3 - \mathbf{p}_2) - 2B_0^3(u_1)(\mathbf{p}_0 - \mathbf{p}_2)}{B_1^3(u_1)} + (\mathbf{p}_1 - \mathbf{p}_2)_\infty.$$

Control point $\mathbf{d}_1$ can be on the surface

$$\mathbf{d}_1(u_1, u_2), \quad u_1 \in (0, 1), \quad u_2 \in (u_1, 1).$$

Asymptotes of its isoparametric lines are as follows. The common asymptote of isoparametric lines with respect to u_1 is

$$\mathbf{p}_1 + \lambda(\mathbf{p}_0 - \mathbf{p}_1), \quad \lambda \in \mathbb{R}.$$

The other set of asymptotes of the same isoparametric lines is the pencil of parallel lines

$$\mathbf{p}_1 + \frac{B_3^3(u_2)(\mathbf{p}_3 - \mathbf{p}_1) - 2B_0^3(u_2)(\mathbf{p}_0 - \mathbf{p}_1)}{B_1^3(u_2)} + \lambda(\mathbf{p}_2 - \mathbf{p}_1), \quad \lambda \in \mathbb{R}, \tag{22}$$

therefore these can be considered as generators of a cylinder.

Asymptotes of isoparametric lines with respect to u_2 form the cylinders

$$\mathbf{p}_2 + \frac{(\mathbf{p}_1 - \mathbf{p}_2) - B_0^3(u_1)(\mathbf{p}_0 - \mathbf{p}_2)}{B_1^3(u_1)} + \lambda(\mathbf{p}_3 - \mathbf{p}_2), \quad \lambda \in \mathbb{R}$$

and

$$\mathbf{p}_2 + \frac{B_3^3(u_1)(\mathbf{p}_3 - \mathbf{p}_2) - 2B_0^3(u_1)(\mathbf{p}_0 - \mathbf{p}_2)}{B_1^3(u_1)} + \lambda(\mathbf{p}_1 - \mathbf{p}_2), \quad \lambda \in \mathbb{R}. \tag{23}$$

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As it is expected, cylinders (22) and (23) coincide. Since the direction of generators of these two cylinders is the same, it is enough to show that the directrix of (23) is on the cylinder (22). The point of directrix of cylinder (23) that corresponds to u_1 is on the $u_2 = u_1$ generator of cylinder (22). The corresponding λ value is

$$\mathbf{p}_1 + \frac{B_3^3(u_1)(\mathbf{p}_3 - \mathbf{p}_1) - 2B_0^3(u_1)(\mathbf{p}_0 - \mathbf{p}_1)}{B_1^3(u_1)}$$

$$+ \lambda(\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{p}_2 + \frac{B_3^3(u_1)(\mathbf{p}_3 - \mathbf{p}_2) - 2B_0^3(u_1)(\mathbf{p}_0 - \mathbf{p}_2)}{B_1^3(u_1)},$$

$$\lambda B_1^3(u_1)(\mathbf{p}_2 - \mathbf{p}_1) = (B_1^3(u_1) - B_3^3(u_1) + 2B_0^3(u_1))(\mathbf{p}_2 - \mathbf{p}_1),$$

$$\lambda = 1 + \frac{2B_0^3(u_1) - B_3^3(u_1)}{B_1^3(u_1)}.$$

Analogous results and properties can be verified for the surface

$$\mathbf{d}_2(u_1, u_2) = \alpha_{02}\mathbf{p}_0 + \alpha_{12}\mathbf{p}_1 + \alpha_{22}\mathbf{p}_2 + \alpha_{32}\mathbf{p}_3 \quad (24)$$

of control point $\mathbf{d}_2$.

6. Interactive Bézier interpolation

Specification of data points $\mathbf{p}_i$ is obvious for a user during the design process. However, specification of nodes u_i is not obvious at all, and there is no universally optimal solution, although these values have a significant effect on the shape of the interpolating curve.

Results of the previous sections enable us to control the shape of the interpolating curve by means of interactive modification of control points instead of specifying scalar values u_i. Modification of a control point is constrained in the sense that the alteration of one node corresponds to the relocation of the control point on a curve (on its path with respect to the node), and the alteration of two nodes corresponds to the relocation of the control point on a surface.

6.1. Interactive modification of a node by a single control point

In case of quadratic ($n = 2$) Bézier interpolation we have only one alterable node u_1, and control point $\mathbf{d}_1$ can be moved along the hyperbolic arc (13). In an interactive design environment, the system displays path (13) and the user modifies the position of $\mathbf{d}_1$ that results the alteration of u_1.

This process can be generalized to higher degree interpolation. Any control point $\mathbf{d}_j$ $(j = 1, 2, \ldots, n-1)$ has a path with respect to any node u_i $(i = 1, 2, \ldots, n-1)$, along which the control point can be moved. (The modification of one control point implies the alteration of the rest of the control points due to the global effect of nodes.) Shape modification can be performed by means of any pair $\mathbf{d}_j, u_i$, however, control point $\mathbf{d}_j$ has the greatest effect on the shape of the curve at parameter value $u = j/n$, thus the preferable choice is that node which is the closest to j/n.

6.2. Interactive specification/modification of nodes by a single control point

In case of cubic Bézier interpolation there are two modifiable nodes u_1 and u_2, consequently control points $\mathbf{d}_1$ and $\mathbf{d}_2$ can be positioned on the surfaces (19) and (24), respectively. These surfaces depend on both u_1 and u_2, therefore the specification of one of the control points determine both u_1 and u_2.

In the case of plane curves, these surfaces degenerate to plane regions. These regions may cover the whole plane or may form two disjoint regions of the plane. This and the shape of these regions depend on the relative positions of data points $\mathbf{p}_i$. The following basic configurations can be distinguished.

- If the quadrilateral $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ is convex, lines $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_3$ are intersecting and this intersection point and data points $\mathbf{p}_0$, $\mathbf{p}_3$ are on different sides of the line $\mathbf{p}_1\mathbf{p}_2$ then regions $\mathbf{d}_1(u_1, u_2)$ and $\mathbf{d}_2(u_1, u_2)$ cover the whole plane.

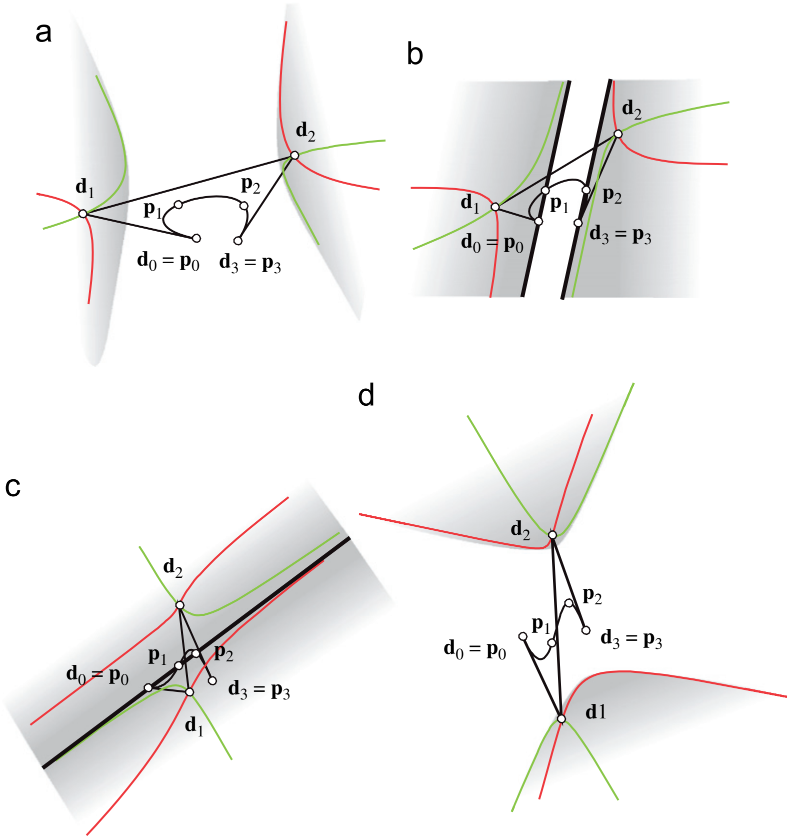

Fig. 3. Basic configurations of control points.

- If the quadrilateral $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ is convex and the lines $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_3$ are parallel then regions $\mathbf{d}_1(u_1, u_2)$ and $\mathbf{d}_2(u_1, u_2)$ form two disjoint halfplanes bounded by the lines $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_3$, respectively (Fig. 3b).
- If data points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ or $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are colinear then regions $\mathbf{d}_1(u_1, u_2)$ and $\mathbf{d}_2(u_1, u_2)$ form two disjoint halfplanes with the common boundary line $\mathbf{p}_1\mathbf{p}_2$ (Fig. 3c).
- If the quadrilateral $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ is convex, lines $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_3$ are intersecting and the intersection point and data points $\mathbf{p}_0, \mathbf{p}_3$ are on the same side of the line $\mathbf{p}_1\mathbf{p}_2$ then regions $\mathbf{d}_1(u_1, u_2)$ and $\mathbf{d}_2(u_1, u_2)$ are disjoint (Fig. 3a).
- If the quadrilateral $\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ is concave then regions $\mathbf{d}_1(u_1, u_2)$ and $\mathbf{d}_2(u_1, u_2)$ are disjoint (Fig. 3d).

7. Conclusions

All interpolation methods require parameterization of the given points, which is a non-intuitive part of geometric design. Instead of further analytical or numerical efforts, in this paper geometric aspects of parameter alteration of interpolation curves are discussed. Beginning from a very general point of view, curves with arbitrary blending functions are considered at first. Then we gradually restricted our study to polynomial curves and finally to Bézier representation. At each level paths of control points and of curve points are described. Fixed points of the control polygon have been found and special parameterization methods have been associated with geometric meaning. Finally, we presented a possible way of interactive interpolation using which control point-based shape alteration can be achieved. This method is intuitive, the designer do not have to deal with numerical data and shape of the curve can similarly be altered as in the case of approximation. Generalization of this technique to higher order curves and to other well-known curve types, especially to spline curves are directions of further research, as well as to find geometric properties of interpolating quadratic Bézier curves with chord length parametrization.

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