

# $C^2$ and $G^2$ continuous spline curves with shape parameters

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**Abstract.** Extensions of the classical B-spline curves by shape parameters are discussed in this paper. Using a method of linear blending, first we extend the abilities of GB-splines, originally defined by uniform shape parameter, to multiple shape parameters. Then two methods, having different types of continuity, are examined in terms of their curvature behavior.

*Key Words:* B-spline curve, shape parameter,  $C^2$  continuity,  $G^2$  continuity

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## 1. Introduction

The classical B-spline curve is defined by its control points but otherwise it is a rigid curve. The generalizations of this curve, which allow some further shape control by shape parameters, are in the forefront of current research due to its importance in applications. Several authors attempted to extend the abilities of the B-spline curve by some shape parameter. Obviously the most known result of this attempt is the NURBS curve (c.f. [9]), but this curve has rational coefficient functions, thus alternative methods tried to incorporate shape parameters into the original, polynomial basis functions. One of the earliest methods in this way is  $\beta$ -spline curve with two global parameters ([3], [1]). Further methods have been provided by direct generalization of B-spline curves as  $\alpha$ B-splines in [8] and [11] and recently as GB-splines in [2]. Some alternative spline curves with shape parameters can be found in [5] and [6].

In this paper we examine and compare two generalizations,  $\alpha$ B-spline curves and GB-spline curves. After defining and discussing the basic properties of these curves in Section 2 and 3, based on a linear blending approach we generalize the GB-spline curves for having multiple shape parameters. In Section 4 we focus on the continuity properties of these curves.

We prove that, with certain restrictions, GB-spline curve with multiple shape parameters has second order geometric ( $G^2$ ) continuity. By comparing the curvature plots of  $\alpha$ B-spline and GB-spline curves we show that, in spite of the general supremacy of  $C^2$  continuity,  $G^2$  continuity sometimes yields better results in geometric modeling, than the classical notion of continuity.

## 2. $\alpha$ B-spline curves

In [8] the  $\alpha$ B-spline curve is introduced as a generalization of the B-spline curve with shape parameters. It is defined by a linear blending method, using the side of the control polygon as target curve, that is modifying the shape parameters the B-spline curve will move towards (or away from) the control polygon. The definition of the  $\alpha$ B-spline curve with a single shape parameter is as follows (for the sake of simplicity, throughout the paper we define only single arc of piecewise curves):

**Definition 1** *Given a sequence of control points  $\mathbf{p}_i, (i = 0, \dots, 3)$ , first consider the cubic uniform B-spline curve*

$$\mathbf{b}(t) = \sum_{i=0}^3 N_i(t) \mathbf{p}_i, \quad t \in [0, 1],$$

where  $N_i(t)$  are the cubic B-spline basis functions. Also consider the line segment  $\mathbf{p}_1\mathbf{p}_2$  parameterized by

$$\mathbf{l}(t) = (1 - s(t))\mathbf{p}_1 + s(t)\mathbf{p}_2, \quad t \in [0, 1]$$

where

$$s(t) = (1 - t^3)^3. \quad (1)$$

Now the arc of the  $\alpha$ B-spline curve  $\mathbf{c}(t)$  is the linear blending of the polygon leg  $\mathbf{p}_1\mathbf{p}_2$  and the original B-spline curve  $\mathbf{b}(t)$  as

$$\mathbf{c}(\alpha, t) = (1 - \alpha)\mathbf{b}(t) + \alpha\mathbf{l}(t), \quad t \in [0, 1], \quad \alpha \in [0, 1]$$

For  $\alpha$ B-spline curves with *multiple* shape parameters an  $\alpha_i$  is associated to the control points  $\mathbf{p}_i, (i = 1, 2)$  and the shape parameter  $\alpha$  will be the function of  $t$ , interpolating the neighboring values  $\alpha_1, \alpha_2$ :

$$\alpha(t) = (1 - s(t))\alpha_1 + s(t)\alpha_2 \quad (2)$$

where  $s(t) = (1 - t^3)^3$  is the same blending function that has been applied for the univariate case.

$\alpha$ B-spline curve with multiple shape parameters has local tension control since  $\alpha_i$  affect only the neighboring arcs.

## 3. GB-spline curves

Another generalization of the classical uniform cubic B-spline curve has been introduced in [2] and is called GB-spline curve. The definition of an arc of a GB-spline curve with shape parameter  $\lambda$  is as follows.

**Definition 2** Given a sequence of control points  $\mathbf{p}_i$ , ( $i = 0, \dots, 3$ ) the arc of the GB-spline curve is

$$\mathbf{c}(\lambda, t) = \sum_{i=0}^3 B_i(\lambda, t) \mathbf{p}_i, \quad \lambda \in [0, \infty), \quad t \in [0, 1],$$

where the GB-spline basis functions are

$$\begin{aligned} B_0(\lambda, t) &= \frac{2}{12 + \lambda} (1 - t)^3 \\ B_1(\lambda, t) &= \frac{1}{12 + \lambda} (2(3 + \lambda)t^3 - 3(4 + \lambda)t^2 + 8 + \lambda) \\ B_2(\lambda, t) &= \frac{1}{12 + \lambda} (-2(3 + \lambda)t^3 + 3(2 + \lambda)t^2 + 6t + 2) \\ B_3(\lambda, t) &= \frac{2}{12 + \lambda} t^3. \end{aligned}$$

At  $\lambda = 0$  the GB-spline curve coincides with the original B-spline curve. Increasing the shape parameter  $\lambda$  the curve arc tends towards the corresponding control leg  $\mathbf{p}_1\mathbf{p}_2$ . In [7] it is proved, that considering a fixed point  $\mathbf{c}(\lambda, t_0)$  of the curve and altering the shape parameter  $\lambda$ , the point will move along a straight line segment. One of the endpoints of this path is  $\mathbf{c}(0, t_0)$ , while the other point is

$$\lim_{\lambda \rightarrow \infty} \mathbf{c}(\lambda, t_0),$$

a point of the control leg  $\mathbf{p}_1\mathbf{p}_2$ . Although it is not obvious from the definition, GB-spline arc  $\mathbf{c}(\lambda, t)$  can also be considered as a linear blending of the original B-spline curve  $\mathbf{b}(t) = \mathbf{c}(0, t)$  and the parameterized control leg  $\mathbf{p}_1\mathbf{p}_2$  (c.f.[7]), similar to the method we have seen in the case of  $\alpha$ B-spline curve. More precisely, one can consider the control leg as

$$\mathbf{l}(t) = s(t)\mathbf{p}_1 + (1 - s(t))\mathbf{p}_2 \quad (3)$$

where

$$s(t) = 2t^3 - 3t^2 + 1. \quad (4)$$

Applying this parametrization, the curve itself can be written in the form

$$\mathbf{c}(\lambda, t) = q(\lambda)\mathbf{c}(0, t) + (1 - q(\lambda))\mathbf{c}(\infty, t),$$

that is

$$\mathbf{c}(\lambda, t) = q(\lambda)\mathbf{b}(t) + (1 - q(\lambda))\mathbf{l}(t)$$

where

$$q(\lambda) = \frac{\lambda}{12 + \lambda}.$$

#### 4. GB-spline curves with multiple shape parameters

The linear blending description of GB-spline curves naturally yields the possibility of introducing multiple shape parameters, similarly to the case of  $\alpha$ B-splines. Applying the basic idea we have seen in Section 2, the GB-spline curve with multiple shape parameters can also be established. Since the linear blending of the  $\alpha$ B-spline and GB-spline curves differ only in the blending functions  $s(t)$ , applying function (4) instead of (1), one can interpolate the newly defined neighboring values of the shape parameters  $\lambda_1, \lambda_2$  just as in eq.(2). That is given the

shape parameters  $\lambda_1, \lambda_2$  associated to the control points  $\mathbf{p}_1, \mathbf{p}_2$ , the shape parameter  $\lambda$  will be the function of  $t$ , interpolating the neighboring values  $\lambda_1, \lambda_2$ :

$$\lambda(t) = (1 - s(t))\lambda_1 + s(t)\lambda_2$$

where  $s(t) = 2t^3 - 3t^2 + 1$ . After some computation the GB-spline curve with multiple shape parameters and its basis functions can be described as follows:

$$\mathbf{g}(\lambda_1, \lambda_2, t) = \sum_{i=0}^3 G_i(\lambda_1, \lambda_2, t)\mathbf{p}_i, \quad \lambda_i \in [0, \infty), \quad t \in [0, 1],$$

where

$$\begin{aligned} G_0 &= \frac{2(1-t)^3}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1} \\ G_1 &= \frac{(\lambda_1 - \lambda_2)(4t^2 - 12t + 9)t^4 + 2(2\lambda_1 - \lambda_2 + 3)t^3 - 3(2\lambda_1 - \lambda_2 + 4)t^2 + 8 + \lambda_1}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1} \\ G_2 &= \frac{-(\lambda_1 - \lambda_2)(2t - 3)^2t^4 - 2(3 + \lambda_1)t^3 + 3(2 + \lambda_1)t^2 + 6t + 2}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1} \\ G_3 &= \frac{2t^3}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1}. \end{aligned}$$

## 5. Continuity at the joint of two arcs

So far we discussed one arc of the curves, defining by 4 control points. However, if the number of control points is  $n > 4$ , that is points  $\mathbf{p}_i, (i = 0, \dots, n-1)$  are given, then the cubic B-spline curve - as well as its generalizations - will contain  $n-2$  arcs, each defined by four consecutive control points. Consider two neighboring arcs of the B-spline curve,  $\mathbf{b}_i(t)$  and  $\mathbf{b}_{i+1}(t)$ . It is a well-known fact (see e.g. [9]), that these arcs have  $C^2$  continuity in their point of contact, that is:

$$\frac{d^r}{dt^r}\mathbf{b}_i(1) = \frac{d^r}{dt^r}\mathbf{b}_{i+1}(0), \quad (r = 0, 1, 2).$$

In case of generalizations of the B-spline curve it is computed in [8], that  $\alpha$ B-spline arcs also possess  $C^2$  continuity at their joint. Moreover, it is proved in [10], that generalizations by linear blending will have  $C^2$  continuity at the point of contact of neighboring arcs if the degree of the blending polynomial  $s(t)$  is at least 5. The only quintic polynomial which fulfills this criteria is (c.f. [4])

$$s(t) = (1-t)^3(6t^2 + 3t + 1).$$

Thus the modified version of  $\alpha$ B-spline curve ([10]) applies this blending function instead of the original function (1), still preserving  $C^2$ -continuity.

This fact immediately yields, that the other generalization, the GB-spline curve has no  $C^2$  continuity at the joint of the consecutive arcs  $\mathbf{g}_i(t)$  and  $\mathbf{g}_{i+1}(t)$ , since the blending function (2) is of degree 3. However it is proved in [2] that GB-spline curve fulfills the criteria of second order *geometric continuity* (or  $G^2$ ) that is, for some constants  $c_1$  and  $c_2$  the following equations hold:

$$\mathbf{g}_i(1) = \mathbf{g}_{i+1}(0) \quad (5)$$

$$\frac{d}{dt}\mathbf{g}_i(1) = c_1 \frac{d}{dt}\mathbf{g}_{i+1}(0) \quad (6)$$

$$\frac{d^2}{dt^2}\mathbf{g}_i(1) = c_1^2 \frac{d^2}{dt^2}\mathbf{g}_{i+1}(0) + c_2 \frac{d}{dt}\mathbf{g}_{i+1}(0). \quad (7)$$

Here we prove that under some restriction the GB-spline curve with multiple shape parameters has also  $G^2$  continuity.

**Theorem 1** *Considering two consecutive arcs  $\mathbf{g}_i(\lambda_i, \lambda_{i+1}, t)$  and  $\mathbf{g}_{i+1}(\lambda_{i+1}, \lambda_{i+2}, t)$  of a GB-spline curve with multiple shape parameters, their joint is  $G^2$  continuous, if  $\lambda_i = \lambda_{i+2}$ , while  $\lambda_{i+1}$  is of arbitrary value.*

*Proof.* By simple substituting one can see that criteria (5) is fulfilled. The first derivatives of the arcs are

$$\begin{aligned} \frac{d}{dt}\mathbf{g}_i(\lambda_i, \lambda_{i+1}, 1) &= \frac{-6}{12 + \lambda_{i+1}}\mathbf{p}_i + \frac{6}{12 + \lambda_{i+1}}\mathbf{p}_{i+2} \\ \frac{d}{dt}\mathbf{g}_{i+1}(\lambda_{i+1}, \lambda_{i+2}, 0) &= \frac{-6}{12 + \lambda_{i+1}}\mathbf{p}_i + \frac{6}{12 + \lambda_{i+1}}\mathbf{p}_{i+2}, \end{aligned}$$

that is, criteria (6) also holds with  $c_1 = 1$ .

The second derivatives of the arcs are computed as

$$\begin{aligned} \frac{d^2}{dt^2}\mathbf{g}_i(\lambda_i, \lambda_{i+1}, 1) &= \left( \frac{12 + 6\lambda_{i+1}}{12 + \lambda_{i+1}} - \frac{12(\lambda_i - \lambda_{i+1})}{(12 + \lambda_{i+1})^2} \right) \mathbf{p}_i + \\ &\quad \left( \frac{-6(2\lambda_{i+1} - \lambda_i + 4)}{12 + \lambda_{i+1}} - \frac{6(\lambda_{i+1} + 8)(\lambda_i - \lambda_{i+1})}{12 + \lambda_{i+1}} \right) \mathbf{p}_{i+1} + \\ &\quad \left( \frac{12}{12 + \lambda_{i+1}} - \frac{12(\lambda_i - \lambda_{i+1})}{(12 + \lambda_{i+1})^2} \right) \mathbf{p}_{i+2} \\ \frac{d^2}{dt^2}\mathbf{g}_{i+1}(\lambda_{i+1}, \lambda_{i+2}, 0) &= \left( \frac{12}{12 + \lambda_1} + \frac{12(\lambda_1 - \lambda_2)}{(12 + \lambda_1)^2} \right) \mathbf{p}_i + \\ &\quad \left( \frac{-12\lambda_1 + 6\lambda_2 - 24}{12 + \lambda_1} + \frac{6(8 + \lambda_1)(\lambda_1 - \lambda_2)}{(12 + \lambda_1)^2} \right) \mathbf{p}_{i+1} + \\ &\quad \left( \frac{12 + 6\lambda_1}{12 + \lambda_1} + \frac{12(\lambda_1 - \lambda_2)}{(12 + \lambda_1)^2} \right) \mathbf{p}_{i+2}. \end{aligned}$$

Since  $c_1 = 1$ , criteria (7) is of the form

$$\frac{d^2}{dt^2}\mathbf{g}_i(1) = \frac{d^2}{dt^2}\mathbf{g}_{i+1}(0) + c_2 \frac{d}{dt}\mathbf{g}_{i+1}(0)$$

which has a unique solution as  $\lambda_i = \lambda_{i+2}$  and  $c_2 = \lambda_{i+1}$ , and this was to be proved.  $\square$

The continuity properties of the two types of spline curves may suggest that  $\alpha$ B-spline curve has smoother shape due to its more rigorous continuity. This fact, however, does not necessary hold as one can see in the following figures of a case study. Applying the GB-spline curve with multiple shape parameters, both spline curves have the same freedom in terms of

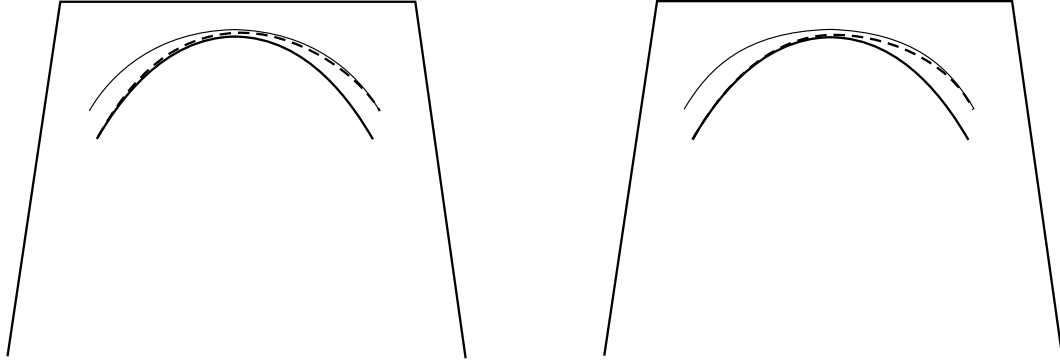


Figure 1: Two versions of spline curves with shape parameters. Thick line: the classical B-spline curve; thin line: the GB-spline curve (left) and the  $\alpha$ B-spline curve (right) with uniform shape parameter; dashed line: the GB-spline curve (left) and the  $\alpha$ B-spline curve (right) with multiple shape parameters.

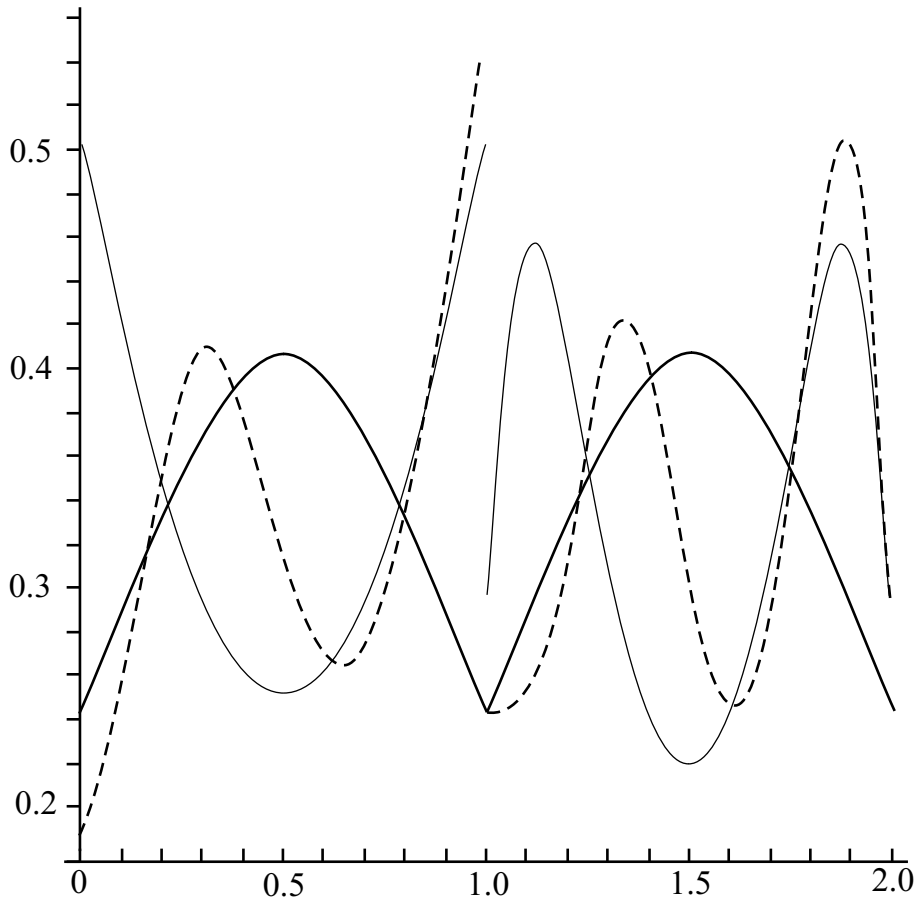


Figure 2: Curvature plot of the curves from Fig.1. (All plots are stretched vertically and plots of the  $\alpha$ B-spline curves are shifted towards right for better visualization)

design and shape control. In Fig.1 one can see two sets of curves defined by the same control polygon. The thick curve is the classical B-spline curve, the thin ones are the generalized spline curves with uniform shape parameter (the GB-spline curve at left and the  $\alpha$ B-spline curve at right) in a visually similar position. The dashed curves are the curves with multiple shape parameters, starting from the classical B-spline curve and ending at the modified curve.

Now as one can observe in Fig.2, although visually there is not much difference between the two curve types, the curvature plot shows, that the strong requirement of  $C^2$  continuity yields unnecessary, heavy oscillations in curvature plot, while the GB-spline curve with  $G^2$  continuity has much smoother plot. This fact somehow shows that weaker conditions sometimes yields more acceptable results in practical geometric modeling.

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## References

- [1] B.A. BARSKY: *Computer graphics and geometric modeling using  $\beta$ -splines*, Springer-Verlag, Berlin (1988).
- [2] Q. GUO: *Cubic GB-spline curves*, Journal of Information and Computational Science **3**, 465–471 (2005).
- [3] B.A. BARSKY, J.C. BEATTY: *Local control of bias and tension in  $\beta$ -splines*, ACM Transactions on Graphics **2**, 109–134 (1983).
- [4] R.H. BARTELS, J.C. BEATTY AND B.A. BARSKY: *An introduction to splines for use in computer graphics and computer modeling*, Morgan Kaufmann, San Francisco, 1987.
- [5] Z. HABIB, M. SAKAI AND M. SARFRAZ: *Interactive Shape Control with Rational Cubic Splines*, International Journal of Computer-Aided Design & Applications, **1**, 709–718 (2004).
- [6] Z. HABIB, M. SARFRAZ AND M. SAKAI: *Rational cubic spline interpolation with shape control*, Computers & Graphics **29**, 594–605 (2005).
- [7] M. HOFFMANN, Y. LI AND G. WANG: *Linear blending and constrained modification of GB-spline curves*, Journal of Zhejiang Univ. Sci. (submitted) (2007).
- [8] K.F. LOE:  *$\alpha B$ -spline: a linear singular blending spline*, The Visual Computer, **12**, 18–25 (1996).
- [9] L. PIEGL, W. TILLER: *The NURBS Book*, Springer-Verlag (1995).
- [10] C.-L. TAI, B.A. BARSKY AND K.F. LOE: *An interpolation method with weights and relaxation parameters*, in: A. COHEN et al. (eds.): *Curves and surface fitting: Saint-Malo*, Vanderbilt Univ. Press, Nashville, 393–402 (2000).
- [11] C.L. TAI, G.J. WANG: *Interpolation with slackness and continuity control and convexity-prservation using singular blending*, Journal of Computational and Applied Mathematics **172**, 337–361 (2004).

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