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Modifying a knot of B-spline curves

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Abstract

The modification of a knot of a B-spline curve of order k generates a family of B-spline curves. We show that an envelope of this family is a B-spline curve defined by the same control polygon, and its order is $k-m$, where m is the multiplicity of the modified knot. Moreover, their arbitrary order derivatives differ only in a multiplier.

Keywords: B-spline curve; Knot modification; Envelope

1. Modifying a knot of single multiplicity

Let us consider the one-parameter family of B-spline curves of order k

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \quad u \in [u_{k-1}, u_{n+1}],\ u_i \in [u_{i-1}, u_{i+1}),\ k > 2 \tag{1}$$

obtained by the modification of the knot u_i of single multiplicity between its neighboring knots. Also consider the B-spline curve

$$\mathbf{b}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l N_l^{k-1}(v), \quad v \in [v_{i-1}, v_i] \tag{2}$$

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of order $k-1$ defined by the same control points $\mathbf{d}_l$, and the knots $v_j = u_j$ if $j < i$ and $v_j = u_{j+1}$ otherwise, i.e., we leave out the knot u_i from the knot vector $\{u_j\}$. Here we prove, that the relation between the derivatives of these two curves at $u = v = u_i$ is

$$\frac{\mathrm{d}^r}{\mathrm{d}v^r}\mathbf{b}(v)\Big|_{v=u_i} = \frac{k-1-r}{k-1}\frac{\mathrm{d}^r}{\mathrm{d}u^r}\mathbf{s}(u,u_i)\Big|_{u=u_i}, \quad r \geqslant 0.$$

(Proofs for the cases $r = 0, 1$ can be found in (Micchelli, 1979; Juhász and Hoffmann, 2001).) To make the two curves compatible, we insert the knot u_i into the knot vector $\{v_j\}$ with Boehm's insertion algorithm (Boehm, 1980). After the conversion of knots from $\{v_j\}$ to $\{u_j\}$, this yields the new representation

$$\mathbf{b}(u) = \sum_{l=i-k+1}^{i-1}\mathbf{d}_l\left(\frac{u_i-u_l}{u_{l+k-1}-u_l}N_l^{k-1}(u) + \frac{u_{l+k}-u_i}{u_{l+k}-u_{l+1}}N_{l+1}^{k-1}(u)\right), \quad u \in [u_{i-1}, u_{i+1}) \tag{3}$$

of curve (2). It is easy to show that $\mathbf{b}(u_i) = \mathbf{s}(u_i, u_i)$, $\forall u_i \in [u_{i-1}, u_{i+1})$.

For the $r > 0$ case we consider the rth derivative of the curve (3)

$$\frac{\mathrm{d}^r}{\mathrm{d}u^r}\mathbf{b}(u) = \sum_{l=i-k+1}^{i-1}\mathbf{d}_l\left(\frac{u_i-u_l}{u_{l+k-1}-u_l}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_l^{k-1}(u) + \frac{u_{l+k}-u_i}{u_{l+k}-u_{l+1}}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_{l+1}^{k-1}(u)\right). \tag{4}$$

The rth derivative of a normalized B-spline basis functions of order k is

$$\frac{k-1-r}{k-1}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_l^k(u) = \frac{u-u_l}{u_{l+k-1}-u_l}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_l^{k-1}(u) + \frac{u_{l+k}-u}{u_{l+k}-u_{l+1}}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_{l+1}^{k-1}(u), \quad k > 1,\ r \geqslant 0,$$

cf. (Butterfield, 1976). Thus the rth derivative of the arc $\mathbf{s}_i(u, u_i)$, $(u \in [u_{i-1}, u_{i+1}))$ with respect to u is

$$\frac{k-1-r}{k-1}\frac{\mathrm{d}^r}{\mathrm{d}u^r}\mathbf{s}_i(u,u_i) = \sum_{l=i-k+1}^{i}\mathbf{d}_l\left(\frac{u-u_l}{u_{l+k-1}-u_l}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_l^{k-1}(u) + \frac{u_{l+k}-u}{u_{l+k}-u_{l+1}}\frac{\mathrm{d}^r}{\mathrm{d}u^r}N_{l+1}^{k-1}(u)\right).$$

The evaluation of this and of Eq. (4) at $u = u_i$ completes the proof.

2. Modifying a knot of higher multiplicity

The above property can be generalized to the case when the modified knot is of multiplicity $m > 1$. In this case the curve (2) becomes

$$\mathbf{b}(v) = \sum_{l=i-k+m}^{i-1}\mathbf{d}_l N_l^{k-m}(v), \quad v \in [v_{i-1}, v_i]$$

on the knots $v_j = u_j$ if $j < i$ and $v_j = u_{j+m}$ otherwise, and the relation between the derivatives is

$$\frac{\mathrm{d}^r}{\mathrm{d}v^r}\mathbf{b}(v)\Big|_{v=u_i} = \frac{\mathrm{d}^r}{\mathrm{d}u^r}\mathbf{s}(u,u_i)\Big|_{u=u_i}\prod_{j=1}^{m}\frac{k-j-r}{k-j}, \quad r \geqslant 0.$$

For a proof of this statement we can show at first, by means of the considerations used in the previous proof, that

$$\frac{\mathrm{d}^r}{\mathrm{d}u^r}\widehat{N}_l^{k-j}(u)\Big|_{u=u_i} = \frac{k-j-r}{k-j}\frac{\mathrm{d}^r}{\mathrm{d}u^r}\widetilde{N}_l^{k-j+1}(u)\Big|_{u=u_i}, \quad j = 1, \ldots, m, \tag{5}$$

where $\widehat{N}_l^{k-j}$ is defined on the knots $\{\ldots, u_{i-1}, u_i = u_{i+1} = \cdots = u_{i+m-j-1}, \ldots\}$ and $\widetilde{N}_l^{k-j+1}$ is on $\{\ldots, u_{i-1}, u_i = u_{i+1} = \cdots = u_{i+m-j}, \ldots\}$. The repeated application of (5) completes the proof.

We mention two corollaries of this property of B-spline curves.

(i) The curve $\mathbf{b}(v)$ of order $k-m$ is an envelope of the family of curves $\mathbf{s}(u, u_i)$ of order k.

(ii) For $k > 3$ spatial curves, $\mathbf{b}(v)$ and $\mathbf{s}(u, u_i)$ have also a common osculating plane at the point of contact. However, their curvatures are different, the relation between them is

$$\kappa_b = \frac{(k-1)(k-m-2)}{(k-2)(k-m-1)}\kappa_s \tag{6}$$

thus $\mathbf{b}(v)$ is a singular curve of the surface $\mathbf{s}(u, u_i)$.

In the rational case we utilize that a rational B-spline curve in $\mathbb{R}^d$ can be described as a central projection of integral B-spline curves in $\mathbb{R}^{d+1}$. It is clear that G^1 continuity contact and the coincidence of the osculating planes remain valid, since these are preserved during central projection. It can also be shown, that the formula (6) holds.

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