COMPUTER-AIDED
DESIGN

Constrained shape modification of cubic B-spline curves by means of knots

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Abstract

The effect of the modification of knot values on the shape of B-spline curves is examined in this paper. The modification of a knot of a B-spline curve of order k generates a one-parameter family of curves. This family has an envelope which is also a B-spline curve with the same control polygon and of order $k - 1$. Applying this theoretical result, three shape control methods are provided for cubic B-spline curves, that are based on the modification of three consecutive knots. The proposed methods enable local shape modifications subject to position and/or tangent constraints that can be specified within well defined limits.
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1. Introduction

B-spline curves and their rational generalizations (NURBS curves) play a central role and are widely used in computer aided design today. These methods are excellent tools in design systems to create new objects, but the modification and shape control of the existing objects are also essential.

The data structure of a B-spline curve of order k is fairly simple, this consists only of control points and knot values. For rational B-spline curves weights have to be specified in addition. Hence, shape control methods can modify such curves only by altering these data. One of the most comprehensive books of this field is Ref. [1], where shape modifications based on control point repositioning and weight modification are described. Further control point-based shape modification methods are discussed in Ref. [2], weight-based modification is described in Refs. [2,3]; and the simultaneous modification of control points and weights are studied in Refs. [4,5]. There are other publications, where shape modification methods are discussed, [6] which presents constraint based manipulations of curves of

arbitrary degree and basis functions. Direct modification of free-form curves by displacement functions is proposed in Ref. [7], which method comprises degree elevation, control point repositioning, knot refinement and removal.

Some aspects of knot modification have also been studied, like in Ref. [8], where the effect of knot variation is examined from numerical point of view. The choice of knot values in curve approximation and interpolation is investigated in several papers, cf. the recently published Ref. [9] and the references therein. Theoretical results in terms of the derivatives of NURBS with respect to a knot have been presented in Ref. [10].

It is an obvious fact, that the modification of the knot vector affects the shape of the curve, the geometric characteristics of this change and its application in shape modification, however, have not been described as yet. In Ref. [11] paths are examined, along which points of B-spline curves move when a knot is altered. It is also shown in Ref. [11], that the one-parameter family of B-spline curves obtained by the modification of a knot has an envelope which is also a B-spline curve. It is known that the modification of one knot makes only slight changes in the shape of a curve. However, a coordinated modification of more knots can result in a considerable change. In Ref. [12] shape control methods are presented for cubic B-spline curves, that utilize paths of points when one knot is

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modified, or two knots are simultaneously modified, and for the rational case when two weights and a knot are simultaneously altered. The idea of a modification when a point of the curve is moved to a specified location is also outlined there.

In this paper we provide shape control tools, that are based on the simultaneous modification of three consecutive knots of integral B-spline curves. Shape modification methods subject to different geometric constraints are discussed, such as move a point of the curve to a specified location; pass through a point with specified tangent; and touch a line at a prescribed point.

There is no doubt, that the most effective shape control tool of integral B-spline curves is control point repositioning. While in the case of rational B-spline curves the change of weights provides a good option, for integral B-spline curves there is no alternative. In this paper we propose a method for integral B-spline curves: the knot alteration. A remarkable advantage of knot modifications over control point repositioning is that the modified curve always remains in the region determined by the convex hull property of the original curve. Thus, knot-based shape control is preferable if we want to keep the curve within this region, or in the case of fine tuning, when the basic form of the curve (the control polygon) is not intended to be changed. The proposed shape modification methods can be implemented in such a way, that the user does not have to take trouble with knots, he/she has to specify only geometric constraints like points and/or lines. These constraints, of course, cannot be chosen arbitrarily. The permissible area of points and the extreme positions of tangent lines are also discussed.

This paper is organized as follows. In Section 2 we describe the theoretic background of the proposed shape modification tools. In Sections 3–5 we discuss the knot-based shape modification methods subject to geometric constraints: pass through a point, touch a line, and move a point of the curve to a location. Finally, the Section 6 contains some concluding remarks.

2. Modifying a knot value

Since in different publications one can find different notations in connection with B-spline curves, we feel necessary to provide the basic definition and notations we are going to use.

Definition 1. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\mathbf{d}_l, \ \ u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order k (degree $k-1$), $(1 < k \leq n+1)$, where $N_l^k(u)$ is the lth normalized B-spline basis function of order k, for the evaluation of which the knots $u_0, u_1, ..., u_{n+k}$ are necessary. Points $\mathbf{d}_l$ are called control points or de Boor points, while the polygon formed by these points is called control polygon.

The jth arc of the B-spline curve of Definition 1 is of the form

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \ \ u \in [u_j, u_{j+1}), \ \ (j = k-1, ..., n)$$

The modification of the knot value u_i between its two neighboring knots, alters the shape of the arcs $\mathbf{s}_j(u)$, $j = (i-k+1, i-k+2, ..., i+k-2)$, since this knot modification affects only the basis functions $N_{i-k}^k(u)$, $N_{i-k+1}^k(u), ..., N_i^k(u)$. By the modification of the knot value u_i, we obtain a one-parameter family of B-spline curves of the form

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \ \ u \in [u_{k-1}, u_{n+1}], \ \ u_i \in [u_{i-1}, u_{i+1})$$

In the quadratic case $(k = 3)$ one can observe that the tangent line of the parabolic arcs at their point of joint is a side of the control polygon, i.e. the linear B-spline curve is a tangent to the quadratic one, defined by the same control polygon. In Ref. [11] the following generalization of this property is proved for arbitrary k

Theorem 2. *Altering a knot value $u_i \in [u_{i-1}, u_{i+1})$ of a B-spline curve $\mathbf{s}(u)$ of order $k(k > 2)$, the one-parameter family of B-spline curves*

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i), \ \ u \in [u_{k-1}, u_{n+1}]$$

has an envelope which is a B-spline curve of order $(k-1)$ and can be written in the form

$$\mathbf{h}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l N_l^{k-1}(v), \ \ v \in [v_{i-1}, v_i]$$

where the knot values

$$v_j = \begin{cases} u_j, & \text{if } j < i \\ u_{j+1}, & \text{otherwise} \end{cases}$$

i.e. from the original knot values $\{u_j\}$ we leave out the ith one. Their points of contact are $\mathbf{h}(u_i) = \mathbf{s}(u_i, u_i)$.

Fig. 1 shows this type of family of cubic B-spline curves along with their envelope. This property can be generalized to higher derivatives as well as to knots of higher multiplicity.

In this paper we concentrate on the manipulation of cubic curves, and we assume, that the knot u_{j+1} is being modified.
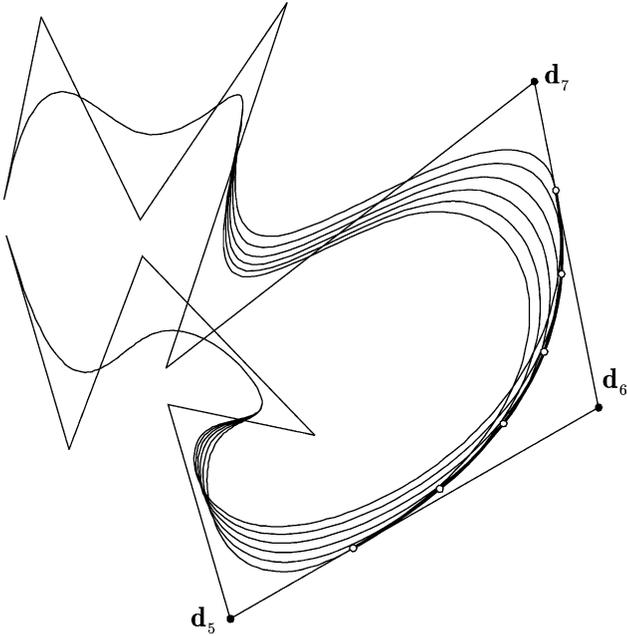
Fig. 1. Members of the family of cubic B-spline curves $\mathbf{s}(u, u_8)$, $u_8 \in [u_7, u_9]$ and their envelope with points of contact.

Thus, the family of curve is

$$\mathbf{s}(u, u_{j+1}) = \sum_{l=0}^{n} \mathbf{d}_l N_l^4(u, u_{j+1}), \quad u \in [u_3, u_{n+1}],$$

$$u_{j+1} \in [u_j, u_{j+2}]$$

with knots $u_0, u_1, \dots, u_{n+4}$, and the envelope is the parabolic arc

$$\mathbf{h}_j(v) = \sum_{l=j-2}^{j} N_l^3(v)\mathbf{d}_l, \quad v \in [v_j, v_{j+1}) \tag{1}$$

with knots $v_{j-2} = u_{j-2}$, $v_{j-1} = u_{j-1}$, $v_j = u_j$, $v_{j+1} = u_{j+2}$, $v_{j+2} = u_{j+3}$, $v_{j+3} = u_{j+4}$.

In the forthcoming sections we show shape modification methods, that use Theorem 2 for constrained shape modification of integral cubic B-spline curves. For brevity, we drop the word integral in the remainder of the paper. The point of these methods is that by means of Theorem 2, we can reduce the manipulation of cubic curves to the manipulation of parabolic arcs, i.e. we can reduce a problem of degree three to a problem of degree two.

3. Pass through a point

An option for constraint-based shape modification is that we specify a point $\mathbf{p}$ through which the modified curve has to pass. We want to perform this modification by the alteration of some knots of the cubic B-spline curve. The process is as follows. We modify the envelope $\mathbf{h}_j(v)$ to pass through the specified point, then we set the appropriate

knots of $\mathbf{s}(u)$ according to Theorem 2 as a result of which the modified $\mathbf{s}(u)$ will touch $\mathbf{h}_j(v)$ at $\mathbf{p}$.

For the sake of solvability, point $\mathbf{p}$ cannot arbitrarily be specified. In the following we determine the region of possible positions of $\mathbf{p}$. The parabolic arc $\mathbf{h}_j(v)$ can be described as a Bézier curve, the control points of which can be obtained from Eq. (1) by the substitutions $v = v_j$ and $v = v_{j+1}$. The resulting control points are

$$\mathbf{b}_0 = (1 - \alpha)\mathbf{d}_{j-1} + \alpha\mathbf{d}_{j-2} \tag{2}$$

$$\mathbf{b}_1 = \mathbf{d}_{j-1}$$

$$\mathbf{b}_2 = (1 - \beta)\mathbf{d}_{j-1} + \beta\mathbf{d}_j$$

with the coefficients

$$\alpha = \frac{v_{j+1} - v_j}{v_{j+1} - v_{j-1}} \tag{3}$$

$$\beta = \frac{v_{j+1} - v_j}{v_{j+2} - v_j}$$

for which $\alpha, \beta \in [0, 1]$ because of the monotonicity of knots. Due to the identity $B_0^2(t) + B_1^2(t) + B_2^2(t) \equiv 1$ of Bernstein polynomials, this Bézier curve can be written in the form

$$\mathbf{b}(t) = \mathbf{b}_1 + B_0^2(t)(\mathbf{b}_0 - \mathbf{b}_1) + B_2^2(t)(\mathbf{b}_2 - \mathbf{b}_1), \quad t \in [0, 1]$$

Utilizing equalities (2) the envelope has the form

$$\mathbf{b}(t) = \mathbf{d}_{j-1} + \alpha B_0^2(t)\mathbf{e}_1 + \beta B_2^2(t)\mathbf{e}_2 \tag{4}$$

in the affine coordinate system

$$\mathbf{d}_{j-1}, \mathbf{e}_1 = \mathbf{d}_{j-2} - \mathbf{d}_{j-1}, \mathbf{e}_2 = \mathbf{d}_j - \mathbf{d}_{j-1} \tag{5}$$

Expression (4) describes a two-parameter family of parabolas, with the family parameters α, β. These parabolas sweep the region the boundary of which is formed by the polygon $\mathbf{d}_{j-2}, \mathbf{d}_{j-1}, \mathbf{d}_j$ and the Bézier curve $\mathbf{b}^1(t)$ determined by these points (it is obtained by the substitution $\alpha = \beta = 1$ into Eq. (4)). We refer to this plane region as Ω, cf. the shaded region in Fig. 2. Parabolas of Eq. (4) can be considered as the affine transforms of the parabola $\mathbf{b}_1(t)$ specifically, they can be obtained by scaling the parabola $\mathbf{b}^1(t)$ with factors (α, β) in the coordinate system (5).

Therefore, if $\mathbf{p} \in \Omega$ a solution to the problem is guaranteed. Now, we examine how many solutions we have, i.e. we determine which parabolas of the form (4) will pass through an arbitrarily chosen point in Ω. $\forall \mathbf{p} \in \Omega$ in the coordinate system (5)

$$\mathbf{p} = \mathbf{d}_{j-1} + x\mathbf{e}_1 + y\mathbf{e}_2$$

If $\mathbf{p}$ is on the boundary of Ω then there is only one parabola, otherwise there are infinite number of parabolas of the type (4) that pass through the point $\mathbf{p}$, i.e. for which

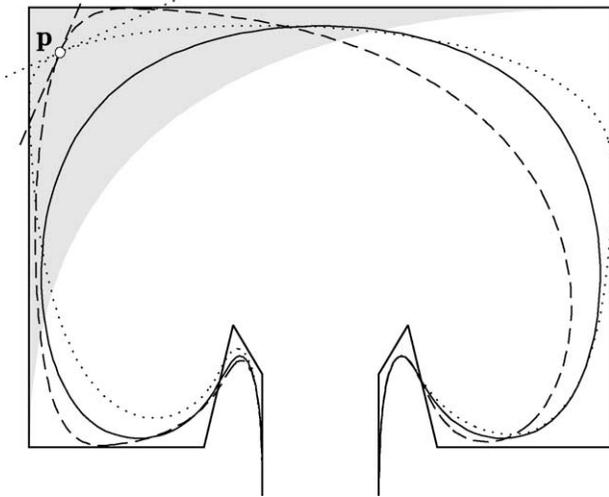$$x = \alpha B_0^2(t), \qquad y = \beta B_2^2(t) \text{ for some } t \in [0, 1] \tag{6}$$

Fig. 2. Permissible positions of **p** (the shaded region), the two extreme positions of parabolas (drawn in dashed) that pass through the point **p**, and the parabola of the specified tangent line **p**$_1$, **p**$_2$.

The extreme positions of these parabolas are at

$$\alpha = 1, \qquad t = 1 - \sqrt{x}, \qquad \beta = \frac{y}{(1 - \sqrt{x})^2}$$

and

$$\beta = 1, \qquad t = \sqrt{y}, \qquad \alpha = \frac{x}{(1 - \sqrt{y})^2}$$

These extreme positions are shown in Fig. 2.

Therefore, $\forall \alpha \in [x/(1 - \sqrt{y})^2, 1] \exists \beta \in [y/(1 - \sqrt{x})^2, 1]$ that the corresponding parabola of Eq. (4) passes through the point **p**. This β is $y/(1 - \sqrt{x/\alpha})^2$. (Of course, it works in the opposite way as well, i.e. $\forall \beta \exists \alpha$). This means, that parabolas of type (4) that pass through the point **p**, form a one-parameter family.

We have to choose an element from this family. This can be done by fixing the parameter value $\tilde{t}$, at which the parabola has to pass through **p**. It follows from Eq. (6), that $\tilde{t}$ can only be between $\sqrt{y}$ and $1 - \sqrt{x}$. This way of choice, however, is not intuitive enough in an interactive design system.

It is more practical to prescribe the tangent of the desired parabola, since this will also be the tangent of the modified cubic B-spline curve at **p**. This tangent can be specified between the tangents of the two extreme positions of parabolas (Fig. 2). Denote the intersection points of the specified tangent and the sides **d**$_{j-2}$, **d**$_{j-1}$ and **d**$_{j-1}$, **d**$_j$ by **p**$_1$ and **p**$_2$, respectively. Then the parameter value $\tilde{t}$ of the parabola for which **b**$(\tilde{t})$ = **p** is

$$\tilde{t} = \frac{|\mathbf{p} - \mathbf{p}_1|}{|\mathbf{p}_2 - \mathbf{p}_1|}$$

Hereby, all the free parameters of Eq. (4) are uniquely determined.

3.1. Back to B-spline representation

Now, we have to return to the B-spline representation of the envelope, then to the cubic B-spline curve **s**(u). This consists of the following steps. With the knowledge of the values α, β and $\tilde{t}$ (**b**$(\tilde{t})$ = **p**) we have to determine

- the B-spline representation of the parabolic arc **b**(t), i.e. the knots v_{j-1}, v_j, v_{j+1} and v_{j+2};
- the parameter value $\tilde{v} \in [v_j, v_{j+1}]$, that corresponds to $\tilde{t} \in [0, 1]$ (with the parameter transformation $\tilde{v} = v_j + \tilde{t}(v_{j+1} - v_j)$);
- the altering knot values u_i of the cubic B-spline curve

$$u_{j-1} = v_{j-1}, \quad u_j = v_j, \quad u_{j+1} = \tilde{v}, \quad u_{j+2} = v_{j+1}, \\ u_{j+3} = v_{j+2}, \tag{7}$$

From these steps above, only the first one is not unique, i.e. the determination of knots $v_i, (i = j - 1, j, ..., j + 2)$, since in Eq. (3) there are two equations and four unknowns. Thus, there are two free parameters left. Any two out of $v_{j-1}, v_j, v_{j+1}, v_{j+2}$ can be fixed, and the system can be solved for the remaining two unknowns. However, in order to minimize the number of modified arcs of the manipulated cubic B-spline curve, we have to solve the system for consecutive knots. We suggest to fix v_{j-1} and v_{j+2}, i.e. to solve the system for v_j and v_{j+1}. The unique solution is guaranteed by the preliminary geometric constraints.

Fig. 3 shows this shape modification for two different tangent lines that pass through the same point **p**. Supposing an interactive design system, the user has to specify the point **p** and either a parameter value $\tilde{t}$ or a tangent line. Permissible position/value of these entities can automatically be displayed by the design system.
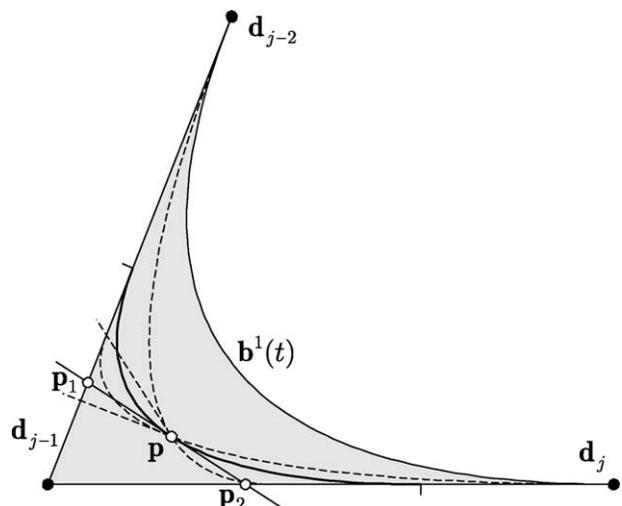

Fig. 3. A cubic B-spline curve and its two modifications that pass through **p** with prescribed tangent lines.

4. Touch a line

Another option for the shape modification of $\mathbf{s}(u)$, is to specify a line the modified curve has to touch, and a point of contact on it. This can be considered as the dual of the problem solved in Section 3.

For the sake of solvability, the specified tangent line has to intersect the sides $\mathbf{d}_{j-2}$, $\mathbf{d}_{j-1}$ and $\mathbf{d}_{j-1}$, $\mathbf{d}_j$ of the control polygon of parabolas (4). We denote these intersection points by $\mathbf{m}_1$ and $\mathbf{m}_2$ (Fig. 4). All points of the straight line segment $\mathbf{m}_1$, $\mathbf{m}_2$ must be in Ω.

In the coordinate system (5)

$$\mathbf{m}_1 = \mathbf{d}_{j-1} + \lambda_1 \mathbf{e}_1, \qquad \mathbf{m}_2 = \mathbf{d}_{j-1} + \lambda_2 \mathbf{e}_2$$

Points of this segment are in Ω, if and only if

$$\lambda_1 + \lambda_2 \leq 1$$

If $\lambda_1 + \lambda_2 = 1$ then there is only one such Bézier curve of the type (4), that touches the specified line segment. This parabola corresponds to the settings $\alpha = \beta = 1$. Otherwise, there is a family of Bézier curves of the type (4), that touch the line segment $\mathbf{m}_1$, $\mathbf{m}_2$. The control points of these Bézier curves are

$$\mathbf{d}_{j-1} + (\lambda_1 + \mu)\mathbf{e}_1, \mathbf{d}_{j-1}, \mathbf{d}_{j-1} + \left(\lambda_2 + \frac{\lambda_1 \lambda_2}{\mu}\right)\mathbf{e}_2 \qquad (8)$$

in the coordinate system (5), where for the parameter μ the inequality

$$\frac{\lambda_1 \lambda_2}{1 - \lambda_2} \leq \mu \leq 1 - \lambda_1$$

has to be fulfilled. Points of contact of these parabolas form the sub-segment $\mathbf{m}_1$, $\mathbf{m}_2$, the endpoints of which are on those parabolas that correspond to the minimum and maximum values of μ. The required point of contact $\mathbf{p}$ can be located on this sub-segment. With the specification of $\mathbf{p}$ we choose an element from the family. The parameter $\tilde{t}$ of this parabola for which $\mathbf{b}(\tilde{t}) = \mathbf{p}$ is

$$\tilde{t} = \frac{|\mathbf{p} - \mathbf{m}_1|}{|\mathbf{m}_2 - \mathbf{m}_1|}$$

The Bézier curve determined by the control points (8) is a parabola of the type (4) with the settings $\alpha = \lambda_1 + \mu$, $\beta = \lambda_2 + (\lambda_1 \lambda_2 / \mu)$. Let the coordinates of $\mathbf{p}$ be x and y in the coordinate system (5), i.e. $\mathbf{p} = \mathbf{d}_{j-1} + x\mathbf{e}_1 + y\mathbf{e}_2$. On the other hand, $\mathbf{p}$ is on the Bézier curve of control points (8). Therefore, $x = B_0^2(\tilde{t})(\lambda_1 + \mu)$, from which the unknown parameter μ in Eq. (8) is

$$\mu = \frac{x}{(1 - \tilde{t})^2} - \lambda_1$$

Herewith, we determined the Bézier representation, i.e. the values α, β, $\tilde{t}$ of the envelope. Now, we have to determine the B-spline representation of the envelope, then we have to set the appropriate knots of $\mathbf{s}(u)$ to achieve the required shape modification. This process is the one, we described in Section 3.1.

Fig. 5 shows this shape modification with two different points of contact on the same tangent line. In a design
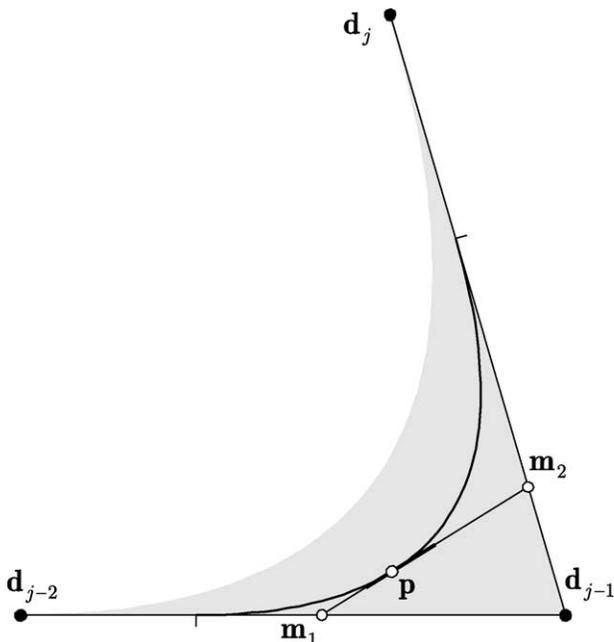

Fig. 4. Permissible positions of tangent segments (the shaded region), the range of points of contact on a tangent segment and a parabolic arc that satisfies the conditions.
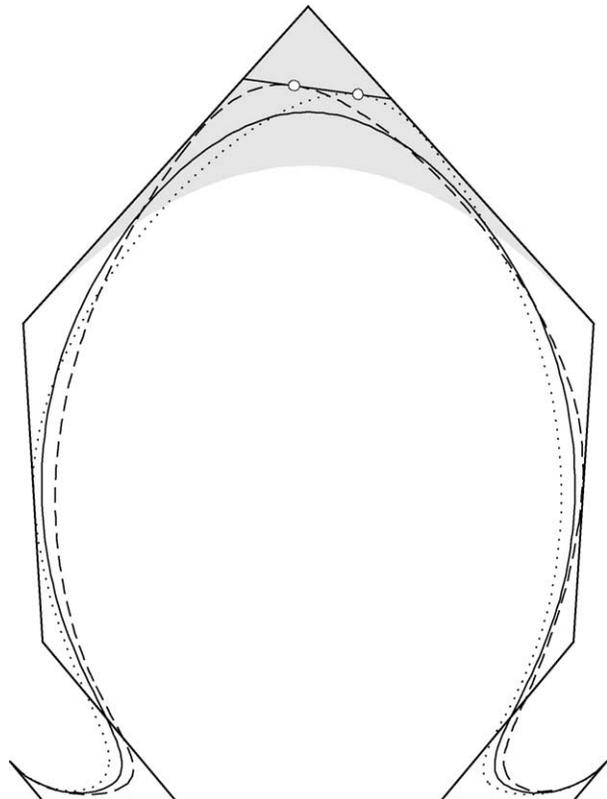

Fig. 5. A cubic B-spline curve and its two modifications that touch the same line.

system, for such a shape modification the user has to specify the tangent line and the point of contact on it. Permissible positions of these entities can automatically be displayed by the design system.

5. Move a point of the curve to a specified location

A frequently used shape modification method is, when the user picks a point on the curve, then species a location where the picked point has to be moved. Hereafter, we assume that for the parameter of the picked point $s(\tilde{u})$, $\tilde{u} \in [u_j, u_{j+2})$ holds. The new location will be denoted by $\mathbf{p}$, and its coordinates in the coordinate system (5) by x and y. We want to perform the $s(\tilde{u}) \rightarrow \mathbf{p}$ shape modification by the alteration of three consecutive knots of the curve $s(u)$.

For the determination of the permissible positions of $\mathbf{p}$ we cannot use what we did in the previous sections. In this case we have to take into account, that in the Bézier representation of the envelope the value $\tilde{t}$ which corresponds to $\tilde{v} = \tilde{u}$, varies with the variation of the knots v_j and v_{j+1}, since $\tilde{t} = (\tilde{v} - v_j)/(v_{j+1} - v_j)$. Therefore, we use the B-spline representation of the envelope.

Utilizing that $N_{j-2}^3(v) + N_{j-1}^3(v) + N_j^3(v) = 1, \forall v \in [v_j, v_{j+1})$, Eq. (1) can be written in the form

$$\mathbf{h}_j(v) = \mathbf{d}_{j-1} + N_{j-2}^3(v)(\mathbf{d}_{j-2} - \mathbf{d}_{j-1}) + N_j^3(v)(\mathbf{d}_j - \mathbf{d}_{j-1})$$

where

$$N_{j-2}^3(v) = \frac{(v_{j+1} - v)^2}{(v_{j+1} - v_{j-1})(v_{j+1} - v_j)}$$

$$N_j^3(v) = \frac{(v - v_j)^2}{(v_{j+2} - v_j)(v_{j+1} - v_j)}$$

Permissible positions of $\mathbf{p}$ form a subset of Ω the bounding curves of which we are going to describe. We allow the change of three pairs of knots. These are (v_{j-1}, v_j), (v_j, v_{j+1}) and (v_{j+1}, v_{j+2}). The corresponding permissible regions of $\mathbf{p}$ will be denoted by Ω_1, Ω_2 and Ω_3, respectively. (In order to minimize the number of altering arcs of $s(u)$, only the change of consecutive knots are allowed.) The boundary of sub regions Ω_i are formed by paths that belong to different extreme positions of the point $\mathbf{h}(\tilde{v})$.

Ω_1 is bounded by three paths. The first path is determined by letting $v_{j-2} = v_{j-1}$ and varying v_j; the second by letting $v_j = \tilde{v}$ and varying v_{j-1} and the third path is determined by letting $v_{j-1} = v_j$ and varying them simultaneously.

Ω_2 is bounded by four paths. The first path is determined by letting $v_{j+1} = v_{j+2}$ and varying v_j; the second by letting $v_{j+1} = \tilde{v}$ and varying v_j the third by letting $v_j = v_{j-1}$ and varying v_{j+1} and the fourth path is determined by letting $v_j = \tilde{v}$ and varying v_{j+1}.

Ω_3 is bounded by three paths. The first path is determined by letting $v_{j+2} = v_{j+3}$ and varying v_{j+1}; the second by letting $v_{j+1} = \tilde{v}$ and varying v_{j+2} and the third path is determined by letting $v_{j+1} = v_{j+2}$ and varying them simultaneously.

These three overlapping regions are shown in Fig. 6.

Thus, if the point $\mathbf{p}$ is in the union of these three regions above, then the solution to the shape modification problem $s(\tilde{u}) \rightarrow \mathbf{p}$ is guaranteed. In such a case, the number of solutions can be 1, 2 or 3 depending on the position of $\mathbf{p}$ with respect to the regions Ω_1, Ω_2 and Ω_3. In order to obtain the solutions, we have to solve the system of equations

$$x = \frac{(v_{j+1} - \tilde{v})^2}{(v_{j+1} - v_{j-1})(v_{j+1} - v_j)} \tag{9}$$

$$y = \frac{(\tilde{v} - v_j)^2}{(v_{j+2} - v_j)(v_{j+1} - v_j)}$$

either for the pair of unknowns (v_{j-1}, v_j) or for (v_j, v_{j+1}) or for (v_{j+1}, v_{j+2}). Only those solutions of Eq. (9) provide solutions to the shape modification problem which fulfills the monotonicity condition

$$v_{j-1} \le v_j \le \tilde{v} \le v_{j+1} \le v_{j+2}$$

as well. Such a solution always exists, when $\mathbf{p}$ is in the region that corresponds to the pair of unknowns, for which the system is solved.

Using the knots $\{v_i\}$, we set the knot values u_i, $(i = j - 1, j...., j + 3)$ according to Eq. (7). The modified curve $s(u)$, that belongs to these knots, will satisfy the requirement
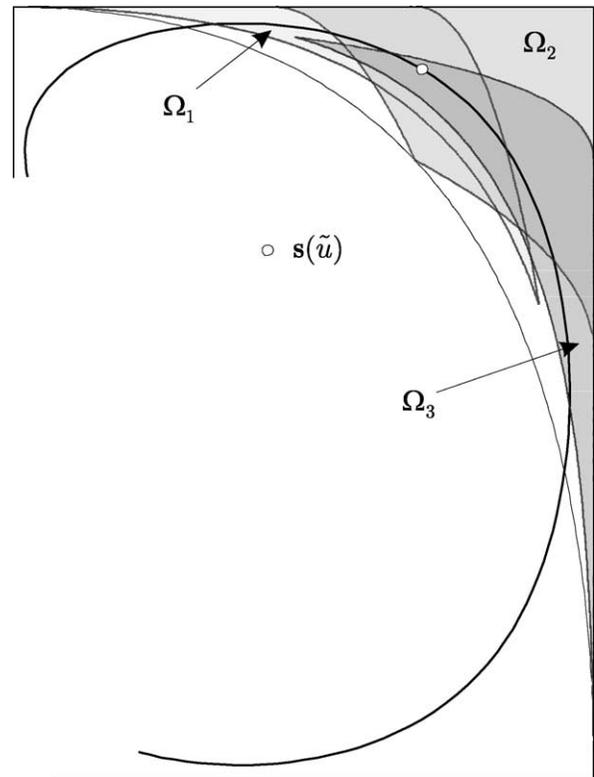

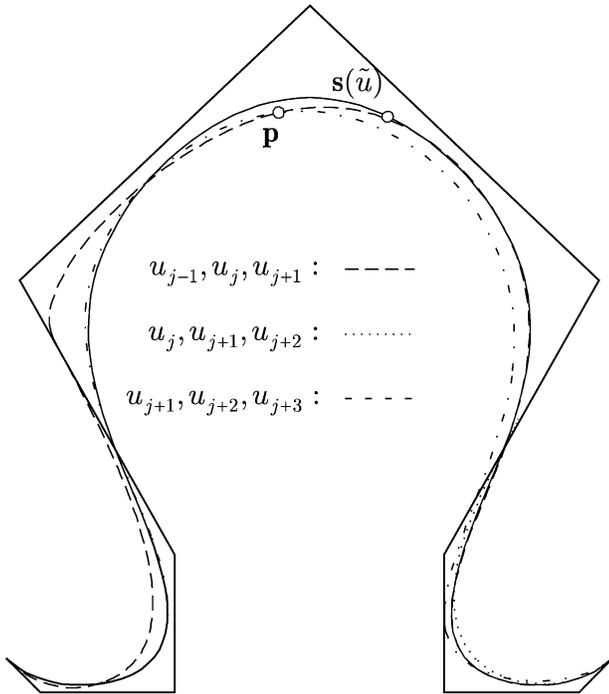Fig. 6. The three overlapping regions within which the picked point can be moved.

Fig. 7. Shape modifications that correspond to the alteration of three different triplets of consecutive knots.

$\mathbf{s}(\tilde{u}) = \mathbf{p}$. Fig. 7 shows such a case, when there are three different solutions.

Certainly, this modification can also be done by control point repositioning. Comparing the two methods, we can see that the repositioning of a control point modifies four arcs of the cubic curve, and there is no restriction for the new position of the picked point $\mathbf{s}(\tilde{u})$. In the case of the proposed knot-based modification method eight arcs of the cubic are modified, and the new position $\mathbf{p}$ is restricted to a relatively small plane region Ω.

On the other hand, the knot based method fairly preserves the character of the curve, cf. Fig. 8, since the modified curve is always remains in the convex hull of the original curve. However, control point repositioning can yield extra inflection and self-intersection points.

In Fig. 9 we show a planar example to demonstrate that even a relatively small translation of $\mathbf{s}(\tilde{u})$ can yield extra inflection points if control point repositioning is applied, see the curve $\mathbf{s}_p(u)$. This happens especially, if the direction of translation is rather tangential than normal to the curve $\mathbf{s}(u)$. However, the curve $\mathbf{s}_k(u)$ obtained by the proposed knot-based method has no extra inflection point. Fig. 10 shows the curvature plots (curvature κ versus arc length s) of these curves.
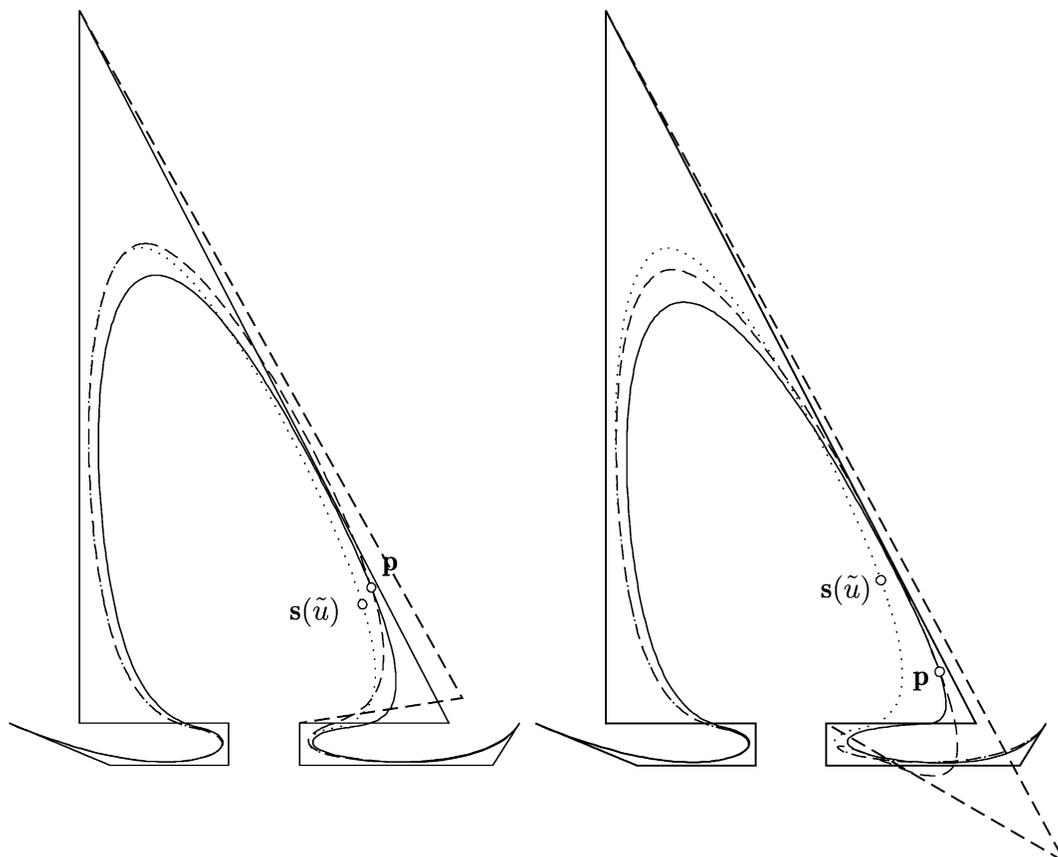

Fig. 8. The point $\mathbf{s}(\tilde{u})$ of a cubic B-spline curve (dotted line) is moved to $\mathbf{p}$ by knot modification (solid line) and by control point repositioning (dashed line).
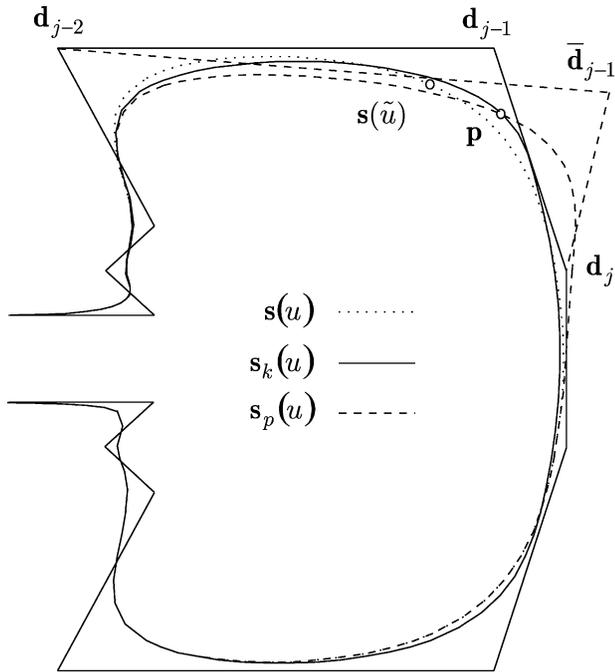
Fig. 9. A cubic B-spline curve $\mathbf{s}(u)$ and its modifications, that move $\mathbf{s}(0.39)$ to a given location $\mathbf{p}$. The curve $\mathbf{s}_p(u)$ obtained by repositioning $\mathbf{d}_{j-1}$ has two extra inflection points, while the curve $\mathbf{s}_k(u)$ obtained by knot alteration remains inside the original convex hull.

This shape modification method can be implemented in such a way, that the user has to pick a point on the curve to be modified, and to specify its new position $\mathbf{p}$. If the point is in more than one regions (Ω_i), then the user must choose either from the solutions provided by the system, or the region for which the problem has to be solved. The regions Ω_i can automatically be displayed by the design system.
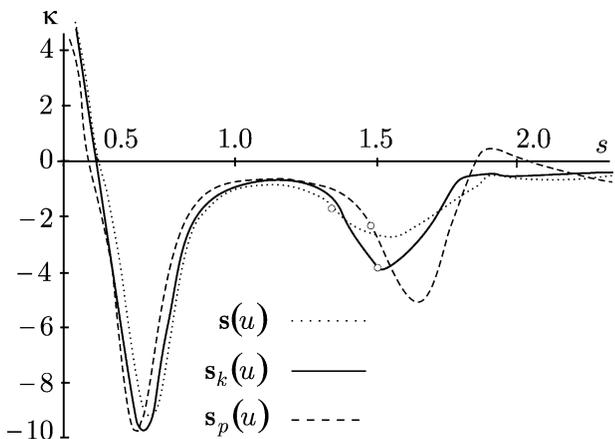

Fig. 10. A part of the curvature plot (curvature κ versus arc length s) of the curves in Fig. 9. One can observe the two extra inflection points of the curve obtained by control point repositioning. Circles indicate the curvature at $\mathbf{p}$ and $\mathbf{s}(\tilde{u})$.

6. Conclusions

Shape modification aspects of knot modification of B-spline curves were presented in this paper. Using a theoretical result about the existence of an envelope of the one-parameter family of B-spline curves, obtained by the modification of a knot value, we presented constrained local shape control methods for integral cubic B-spline curves. These methods are based on the alteration of consecutive knot values of the curve. A definite advantage of knot modification over the repositioning of control points is that the modified curve always remains within the convex hull of the original curve. Thus, knot-based modification may be preferred if shape preservation is of importance. However, the permissible positions of the geometrical constraints are limited. Three options were described for shape modification of cubic B-spline curves, such as

- pass through a point with a prescribed tangent direction,
- touch a line at a prescribed point of contact,
- pass through a point at a prescribed parameter value.

Using these methods, users do not have to deal with knots, they just have to specify geometric constraints.

Shape modification methods, we have described for cubic B-spline curves, can be generalized to B-spline curves of arbitrary degree, if we generalize Theorem 2 to higher multiplicity knots. Thus, if we have a B-spline curve $\mathbf{s}(u)$ of order $k > 3$, and we want to perform a shape modification of the type described in Sections 3–5 with the modification of the knot u_{j+1}, we have to insert u_{j+1} repeatedly with Böhm's knot insertion algorithm [13] until its multiplicity becomes $k - 3$. Therefore, the envelope of the family of curves $\mathbf{s}(u, u_{j+1})$ will be a quadratic B-spline for which considerations described in the previous sections are valid. Nevertheless, one has to take into account that knot insertion yields a new control polygon the concerned part of which is closer to the curve than the original one, consequently, the region of change (permissible position of $\mathbf{p}$) becomes smaller.

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