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Paths of C-Bézier and C-B-spline curves [1]

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Abstract

C-Bézier and C-B-spline curves, as the trigonometric extensions of cubic uniform spline curves are well-known in geometric modeling. These curves depend on a shape parameter $\alpha \in (0, \pi]$ about what the only fact we know is that $\alpha \to 0$ yields the cubic polynomial curves. The geometric effect of the alteration of this parameter is discussed in this paper by the help of relative parametrization and linear approximation.

Key words: C-Bézier curves, C-B-spline curves, paths

1 Introduction

In the last decade several new types of spline curves and surfaces have been introduced to CAGD. C-curves are extensions of the widely used cubic spline curves and are introduced by Zhang (1996) applying the basis $\sin t, \cos t, t, 1$. In the case of C-B-splines this extension coincides with the helix splines defined by Pottmann and Wagner (2002). These tools provide exact representations of several important curves and surfaces such as the circle and the cylinder (Zhang, 1996), the ellipse (Zhang, 1999), the sphere (Morin et al., 1999), the cycloid and the helix (Mainar et al., 2001). Further properties of C-curves have been studied by Mainar and Pena (2002) and by Yang and Wang (2004).

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C-curves are all defined on the interval $t \in [0, \alpha]$, where $\alpha \in (0, \pi]$ is a given real number. Since α appears in all the basis functions, it heavily affects the shape of the curve. While it is already proved (Zhang, 1996), that the limiting case $\alpha \to 0$ is a cubic polynomial curve, the effects of the modification of α have not been described yet. The aim of this paper is to give a geometric interpretation of the change of α for C-Bézier and C-B-spline curves.

Modifying one or more data of a given spline curve, the points of the curve will move on certain curves called paths. For example moving one of the control points of a Bézier or B-spline curve these paths will be parallel line segments, while changing a weight of a NURBS curve points of this curve will move towards the specified control point along line segments (Piegl and Tiller, 1995). Alteration of a knot value of a non-uniform B-spline curve yields well-defined rational curves as paths (Juhász and Hoffmann, 2003). If the parameter α of a C-curve is altered, the points of the curve obviously change their positions as well. In this paper these paths of C-Bézier and C-B-spline curves will be discussed. These paths can closely be approximated by lines and have some nice geometric properties which may yield to a better understanding of the role of α in terms of the shape of these curves.

2 Paths of C-Bézier curves and their extensions

Consider the C-Bézier curve (c.f. Zhang (1996)):

$$\mathbf{b}(t, \alpha) = \sum_{i=0}^{3} Z_i(t, \alpha)\mathbf{p}_i, \quad t \in [0, \alpha], \alpha \in (0, \pi]$$

where the basis functions are defined as:

$$\begin{aligned}
M &= \begin{cases} 1 & \text{if } \alpha = \pi, \\ \frac{\sin(\alpha)}{\alpha - 2\frac{\alpha - \sin(\alpha)}{1-\cos(\alpha)}} & \text{otherwise} \end{cases} \\
Z_0(t, \alpha) &= \frac{(\alpha - t) - \sin(\alpha - t)}{\alpha - \sin(\alpha)} \\
Z_1(t, \alpha) &= M\left(\frac{1-\cos(\alpha - t)}{1-\cos(\alpha)} - \frac{(\alpha - t) - \sin(\alpha - t)}{\alpha - \sin(\alpha)}\right) \\
Z_2(t, \alpha) &= M\left(\frac{1-\cos(t)}{1-\cos(\alpha)} - \frac{t - \sin(t)}{\alpha - \sin(\alpha)}\right) \\
Z_3(t, \alpha) &= \frac{t - \sin(t)}{\alpha - \sin(\alpha)}.
\end{aligned} \tag{1}$$

We would like to describe the movement of a single point of the curve as the parameter α changes. Altering this parameter we receive a family of C-Bézier curves with family parameter α. Due to the changing domain of definition there is not much sense to examine a point of these curves with fixed parameter t. Instead we consider the point at each curve associated to the parameter $(\alpha/ratio)$, where $ratio \in [1, \infty)$ is a fixed value. This parameter changes from curve to curve but if the domain of definition $[0, \alpha]$ would be normalized to $[0, 1]$ for each α, then the specified parameter $(\alpha/ratio)$ would have been transformed to the constant value $(1/ratio)$. This way we can define the *relative α-paths* of the family of C-Bézier curves:

$$\mathbf{s}(\alpha, ratio) = \sum_{i=0}^{3} Z_i(\alpha/ratio, \alpha)\mathbf{p}_i, \quad \alpha \in (0, \pi]; ratio \in [1, \infty)$$

where α is the running parameter along the path, while $ratio$ is the parameter of the path among the family of paths (see Fig.1).

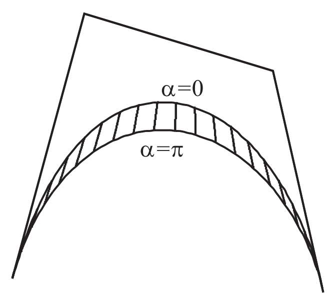

Fig. 1. Two C-Bézier curves defined by the same control polygon and their relative α-paths

Note, that the basis functions of the original C-Bézier curve are symmetric in t for the parameter $t = \alpha/2$, thus the relative α-paths also have a symmetric property in $ratio$ for the parameter $ratio = 2$. The relative α-path associated to $ratio = 2$ can be described by the functions

$$\begin{aligned} Z_0(\alpha/2, \alpha) = Z_3(\alpha/2, \alpha) &= \frac{(\alpha/2) - \sin(\alpha/2)}{\alpha - \sin(\alpha)} \\ Z_1(\alpha/2, \alpha) = Z_2(\alpha/2, \alpha) &= M\left(\frac{1 - \cos(\alpha/2)}{1 - \cos(\alpha)} - \frac{(\alpha/2) - \sin(\alpha/2)}{\alpha - \sin(\alpha)}\right) \end{aligned}$$

which obviously yields that this path is a part of the line connected the midpoints of $\mathbf{p}_0\mathbf{p}_3$ and $\mathbf{p}_1\mathbf{p}_2$. Paths associated to $\alpha \neq 2$ are not lines as one can easily observe by the mathematical extension of the paths (see Fig 2.). This extension is defined by the points

$$\mathbf{s}(\alpha, ratio) = \sum_{i=0}^{3} Z_i(\alpha/ratio, \alpha)\mathbf{p}_i, \quad ratio \in [1, \infty)$$

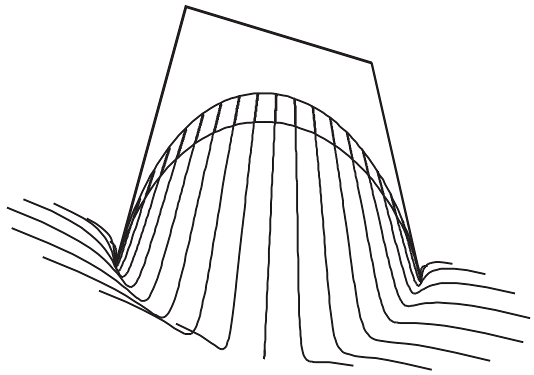

Fig. 2. Extension of the paths for $\alpha \geq \pi$

for $\alpha \geq \pi$. We have to emphasize that these points do not belong to any C-Bézier curves and the substitution of these values of α is merely a mathematical extension. Similar extension have been successfully used for paths of B-spline curves by Hoffmann and Juhász (2004).

3 Approximate lines of the paths

The paths, as we have seen are not lines, but in the original interval $\alpha \in (0, \pi]$ they can closely be approximated by lines. The approximate line of the path $\mathbf{s}(\alpha, ratio)$ can be defined by the joint segment of the point $\mathbf{s}(\pi, ratio)$ and $\mathbf{s}(0, ratio)$ (more precisely, since α cannot be equal to 0, we consider the point obtained by $\alpha \to 0$ in this latter case).

If $\alpha = \pi$ and $t = \pi/ratio$, then

$$\begin{aligned} M &= 1 \\ Z_0(\pi/ratio, \pi) &= \frac{(\pi - \pi/ratio) - \sin(\pi/ratio)}{\pi} \\ Z_1(\pi/ratio, \pi) &= \frac{1 + \cos(\pi/ratio)}{2} - \frac{(\pi - \pi/ratio) - \sin(\pi/ratio)}{\pi} \\ Z_2(\pi/ratio, \pi) &= \frac{1 - \cos(\pi/ratio)}{2} - \frac{\pi/ratio - \sin(\pi/ratio)}{\pi} \\ Z_3(\pi/ratio, \pi) &= \frac{\pi/ratio - \sin(\pi/ratio)}{\pi}. \end{aligned} \tag{2}$$

If $\alpha \to 0$, then from equations (1) we obtain the following limits (see Zhang (1996)):

$$
\begin{aligned}
Z_{0\,\lim}(ratio) &= \frac{(ratio-1)^3}{ratio^3} \\
Z_{1\,\lim}(ratio) &= 3\frac{(ratio-1)^2}{ratio^3} \\
Z_{2\,\lim}(ratio) &= 3\frac{ratio-1}{ratio^3} \\
Z_{3\,\lim}(ratio) &= \frac{1}{ratio^3}.
\end{aligned} \tag{3}
$$

Finally joining the point $\sum_{i=0}^{3} Z_i(\pi/ratio, \pi)\mathbf{p}_i$ and the point $\sum_{i=0}^{3} Z_{i\,\lim}(ratio)\mathbf{p}_i$ we obtain a family of lines with family parameter *ratio* (c.f. Fig. 3):

$$
\mathbf{e}(t,ratio) = \sum_{i=0}^{3}(tZ_i(\pi/ratio,\pi) + (1-t)Z_{i\,\lim}(ratio))\mathbf{p}_i
$$
$$
t \in (-\infty,\infty); ratio \in [1,\infty).
$$

Without specifying the control polygon the difference between the paths and their approximate lines can be evaluated by the following way: consider the path $\mathbf{s}(\alpha, ratio_0)$ which is approximated by the line $\mathbf{e}(t,ratio_0)$. The points of this line between the points $\sum_{i=0}^{3} Z_i(\pi/ratio_0, \pi)\mathbf{p}_i$ and $\sum_{i=0}^{3} Z_{i\,\lim}(ratio_0)\mathbf{p}_i$ can be described by the barycentric coordinates of these two endpoints. Consider a fixed point of the path $\mathbf{s}(\alpha_0, ratio_0)$ for which the appropriate point of the line segment is the point with barycentric coordinates $((\alpha_0/\pi)^2, 1-(\alpha_0/\pi)^2)$. Now we can calculate the difference between the points

$$
\mathbf{s}(\alpha_0, ratio_0) \text{ and } \mathbf{e}\left((\alpha_0/\pi)^2, ratio_0\right).
$$

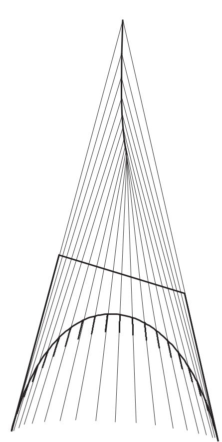

Fig. 3. Approximate lines of the paths

Denoting the coefficient functions of $\mathbf{e}\left((\alpha_0/\pi)^2, ratio_0\right)$ by

$$E_i(\alpha_0, ratio_0) := \left(\left(\frac{\alpha_0}{\pi}\right)^2 Z_i(\pi/ratio_0, \pi) + \left(1 - \left(\frac{\alpha_0}{\pi}\right)^2\right) Z_{i\,\mathrm{lim}}(ratio_0)\right),$$

these points can be written as:

$$\begin{aligned} \mathbf{s}(\alpha_0, ratio_0) &= \sum_{i=0}^{3} Z_i(\alpha_0/ratio_0, \alpha_0)\mathbf{p}_i \\ \mathbf{e}\left((\alpha_0/\pi)^2, ratio_0\right) &= \sum_{i=0}^{3} E_i(\alpha_0, ratio_0)\mathbf{p}_i. \end{aligned}$$

The difference can be described by the standard deviation function

$$stdev(\alpha_0, ratio_0) = \sum_{i=0}^{3} \left(Z_i(\alpha_0/ratio_0, \alpha_0) - E_i(\alpha_0, ratio_0)\right)^2$$

where $\alpha_0 \in (0, \pi], ratio_0 \in [1, \infty)$. The graph of this function can be seen in Fig. 4.

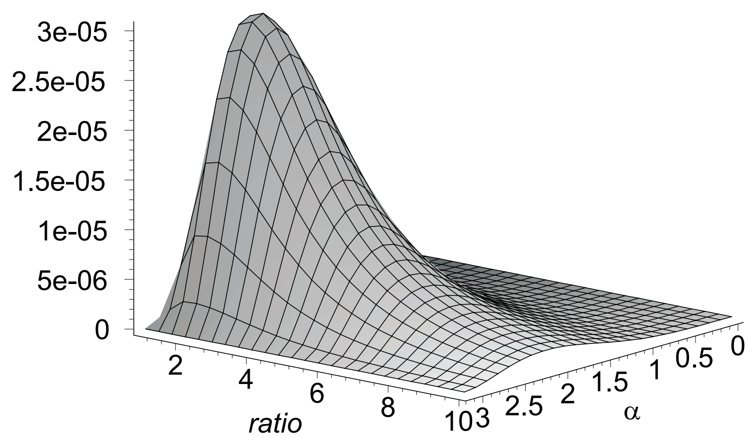

Fig. 4. The graph of the difference between the paths and their approximate lines

As one can see, the maximum deviation is 0.00003. Due to the symmetry mentioned above this function is also symmetric for the value $ratio = 2$ in a way, that it has the same behavior in the interval $[2, 1)$ as in $[2, \infty)$ by the substitution $ratio = ratio/(ratio - 1)$. To provide a geometric viewpoint of this deviation, we can consider the difference between the points $\mathbf{s}(\alpha_0, ratio_0)$ and $\mathbf{e}\left((\alpha_0/\pi)^2, ratio_0\right)$ as

$$\mathbf{e}\left((\alpha_0/\pi)^2, ratio_0\right) = \mathbf{s}(\alpha_0, ratio_0) + \sum_{i=0}^{3} \triangle_i \mathbf{p}_i,$$

where

$$\triangle_i = E_i(\alpha_0, ratio_0) - Z_i(\alpha_0/ratio_0, \alpha).$$

From the graph of the functions $\triangle_i$ in Fig. 5 one can observe, that these functions are almost symmetric for $i = 0, 1$ and $i = 1, 2$. Moreover, it is easy to calculate that their sum equals 0 for all α and *ratio*. Thus the difference is overestimated if the functions $\triangle_i$ are replaced by their overall extremum 0.0035. This yields the expression

$$\mathbf{e}\left((\alpha_0/\pi)^2, ratio_0\right) = \mathbf{s}(\alpha_0, ratio_0) + 0.0035\left((\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_3 - \mathbf{p}_2)\right).$$

from which one can see how the deviation is related to the legs of the control polygon.

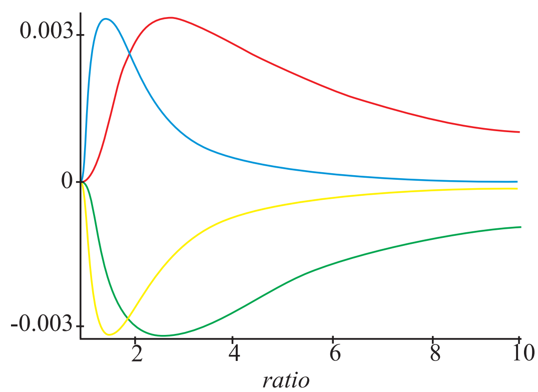

Fig. 5. The graph of the functions $\triangle_i$ at $\alpha = 2$ ($i = 0$:red; $i = 1$:green; $i = 2$:yellow; $i = 3$:blue).

4 Intersection of symmetric lines

To describe further properties of these lines we can consider the "symmetric" pairs of lines that is, the lines $\mathbf{e}(t,ratio)$ and $\mathbf{e}(t,ratio/(ratio-1))$ ($ratio \in (1,2]$) which approximate the paths of the points associated to the relative parameters $\alpha/ratio$ and $\alpha - \alpha/ratio$. If the four control points are coplanar, then the intersection points of these symmetric pairs of lines form a curve with parameter *ratio* (see Fig. 6). The starting point of these section curves ($ratio \to 1$) is always the intersection points of the legs $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_3$, while the endpoint ($ratio \to 2$) of the intersection curves is on the line (and on the relative path) associated to the parameter $\alpha/2$. This latter endpoint can be found by a limit: for each parameter *ratio* one can compute the parameters t_0 and t_1 of the intersection point of the two lines $\mathbf{e}(t_0,ratio) = \mathbf{e}(t_1,ratio/(ratio-1))$, from which, applying the limit $ratio \to 2$

$$
\begin{aligned}
num =& (-4+\pi)\,\|\mathbf{p}_0 \times \mathbf{p}_2\| - \pi\,\|\mathbf{p}_0 \times \mathbf{p}_1\| + 4\,\|\mathbf{p}_0 \times \mathbf{p}_3\| + \\
& +(-4+\pi)\,\|\mathbf{p}_1 \times \mathbf{p}_3\| - \pi\,\|\mathbf{p}_2 \times \mathbf{p}_3\| + (4-2\pi)\,\|\mathbf{p}_1 \times \mathbf{p}_2\| \\
denom =& (-4+\pi)\,\|\mathbf{p}_0 \times \mathbf{p}_2\| + (3-\pi)\,\|\mathbf{p}_0 \times \mathbf{p}_1\| + \|\mathbf{p}_0 \times \mathbf{p}_3\| + \\
& +(-4+\pi)\,\|\mathbf{p}_1 \times \mathbf{p}_3\| + (7-2\pi)\,\|\mathbf{p}_1 \times \mathbf{p}_2\| + (3-\pi)\,\|\mathbf{p}_2 \times \mathbf{p}_3\| \\
t_{\lim} =& \frac{num}{denom}.
\end{aligned}
$$

Thus the endpoint of the intersection curve is $\mathbf{e}(t_{\lim},2)$.

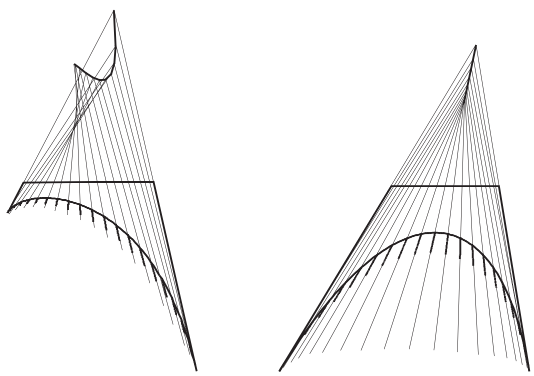

Fig. 6. Intersection curves of the symmetric lines: if $\mathbf{p}_0\mathbf{p}_3$ is parallel to $\mathbf{p}_1\mathbf{p}_2$, then the intersection curve is a line segment

The shape of this intersection curve naturally depends on the positions of the control points. A special case is demonstrated by the following result.

Theorem 1 *The intersection curve of the symmetric lines is a straight line segment if $\mathbf{p}_0\mathbf{p}_3$ is parallel to $\mathbf{p}_1\mathbf{p}_2$. Furthermore, the line segment is just on the line connecting the midpoint of $\mathbf{p}_0\mathbf{p}_3$ and the midpoint of $\mathbf{p}_1\mathbf{p}_2$.*

Proof: Considering the fact that the C-Bézier curve, the symmetric lines and the midpoint of the segments are all preserved by an affine transformation (for the affine invariance of the C-Bézier curve see Zhang (1996)), we prove the result in a special case using the coordinate system given in Fig. 7.

The two pairs of points of the symmetric lines which replace the relative α-paths are on the curve $\alpha = 0$ and $\alpha = \pi$ respectively, so we can compute the two curves $\mathbf{b}_{\mathbf{lim}}(ratio) = lim_{\alpha\to 0}\mathbf{b}(\alpha/ratio, \alpha)$ and $\mathbf{b}(\pi/ratio, \pi)$ by the expressions (2) and (3).

Using the abbreviation $r = ratio$ the considered points are

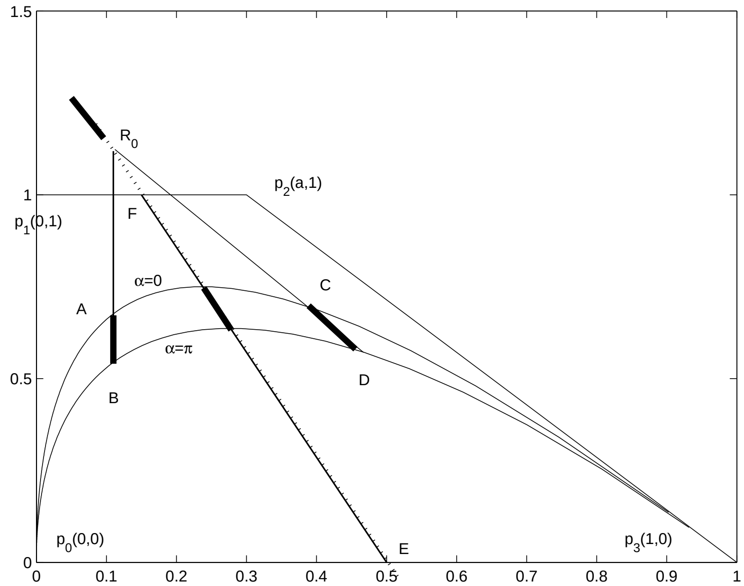

Fig. 7. The special case and the coordinate system

$$\mathbf{A} = \sum_{i=0}^{3} Z_{ilim}(r)\mathbf{p}_i$$
$$\mathbf{B} = \sum_{i=0}^{3} Z_i(\pi/r, \pi)\mathbf{p}_i$$
$$\mathbf{C} = \sum_{i=0}^{3} Z_{ilim}(\frac{r}{r-1})\mathbf{p}_i$$
$$\mathbf{D} = \sum_{i=0}^{3} Z_i(\pi/\frac{r}{r-1}, \pi)\mathbf{p}_i.$$

Applying the coordinate system of Fig. 7 the coordinates of these points can be written as

$$\mathbf{A} = \left(\frac{3(r-1)a+1}{r^3}, \frac{3(r-1)^2+3(r-1)}{r^3}\right)$$
$$\mathbf{B} = \left(\frac{a}{2} - \frac{a}{2}\cos\frac{\pi}{r} - \frac{a}{r} + \frac{a}{\pi}\sin\frac{\pi}{r} + \frac{1}{r} - \frac{1}{\pi}\sin\frac{\pi}{r}, \frac{2}{\pi}\sin\frac{\pi}{r}\right)$$
$$\mathbf{C} = \left(\frac{3(r-1)^2a+(r-1)^3}{r^3}, \frac{3(r-1)^2+3(r-1)}{r^3}\right)$$
$$\mathbf{D} = \left(\frac{a}{2}\cos\frac{\pi}{r} - \frac{a}{2} + \frac{a}{r} + \frac{a}{\pi}\sin\frac{\pi}{r} + 1 - \frac{1}{r} - \frac{1}{\pi}\sin\frac{\pi}{r}, \frac{2}{\pi}\sin\frac{\pi}{r}\right)$$

So the intersection point of **AB** and **CD** is $\mathbf{R}(\lambda) = \lambda\mathbf{A} + (1-\lambda)\mathbf{B}$, where

$$\lambda = \frac{ar^3\cos\frac{\pi}{r} - ar^3 + 2ar^2 + r^3 - 2r^2}{ar^3\cos\frac{\pi}{r} - ar^3 + 2ar^2 + (r^2 - 3r + 2)(1 - 3a)}, \quad r \in [1, 2].$$

Denote the midpoint of $\mathbf{p}_0\mathbf{p}_3$ by $\mathbf{E}$ and the midpoint of $\mathbf{p}_1\mathbf{p}_2$ by $\mathbf{F}$. The reciprocal of slope of $\mathbf{EF}$ is

$$\frac{1}{k_{\mathbf{EF}}} = \frac{a-1}{2}$$

We can also compute the reciprocal of slope of $\mathbf{ER}(\lambda)$:

$$\frac{1}{k_{\mathbf{ER}(\lambda)}} = \frac{\lambda(\frac{3(r-1)a+1}{r^3}) + (1-\lambda)(\frac{a}{2} - \frac{a}{2}\cos\frac{\pi}{r} - \frac{a}{r} + \frac{a}{\pi}\sin\frac{\pi}{r} + \frac{1}{r} - \frac{1}{\pi}\sin\frac{\pi}{r}) - \frac{1}{2}}{\lambda(\frac{3(r-1)^2+3(r-1)}{r^3}) + (1-\lambda)(\frac{2}{\pi}\sin\frac{\pi}{r})}.$$

By computing the common factor of the denominator and the numerator, and simplifying the expression we obtain

$$\frac{1}{k_{\mathbf{ER}(\lambda)}} = \frac{a-1}{2} = \frac{1}{k_{\mathbf{EF}}}$$

which completes the proof. □

5 Paths of C-B-spline curves and their approximate lines

C-B-spline curves are also introduced by Zhang (1996) who also provided the following formula of this curve in (Zhang, 1999)(for the sake of simplicity here we consider only four control points with a single C-B-spline arc):

$$\mathbf{b}(t, \alpha) = \sum_{i=0}^{3} B_i(t, \alpha)\mathbf{p}_i, \quad t \in [0, \alpha], \alpha \in (0, \pi]$$

where the basis functions are defined as:

$$
\begin{aligned}
B_0(t,\alpha) &= \frac{(\alpha - t) - \sin(\alpha - t)}{2\alpha(1-\cos\alpha)} \\
B_3(t,\alpha) &= \frac{t - \sin t}{2\alpha(1-\cos\alpha)} \\
B_1(t,\alpha) &= B_3(t,\alpha) - 2B_0(t,\alpha) + \frac{2(\alpha - t)(1-\cos\alpha)}{2\alpha(1-\cos\alpha)} \\
B_2(t,\alpha) &= B_0(t,\alpha) - 2B_3(t,\alpha) + \frac{2t(1-\cos\alpha)}{2\alpha(1-\cos\alpha)}.
\end{aligned}
\tag{4}
$$

Relative α-paths $\mathbf{s}(\alpha, ratio)$ of C-B-spline curves can analogously be defined to the case of C-Bézier curves. Mathematical extension of these paths for $\alpha \geq \pi$ is also similar to that one we have seen in the previous section (see Fig. 8). The path associated to $ratio = 2$ is a line again, due to the equalities

$$
\begin{aligned}
\mathbf{B}_0 = \mathbf{B}_3 &= \frac{2\sin(\alpha/2) - \alpha}{4\alpha(\cos\alpha - 1)} \\
\mathbf{B}_1 = \mathbf{B}_2 &= \frac{-2\sin(\alpha/2) - \alpha + 2\alpha\cos\alpha}{4\alpha(\cos\alpha - 1)}.
\end{aligned}
$$

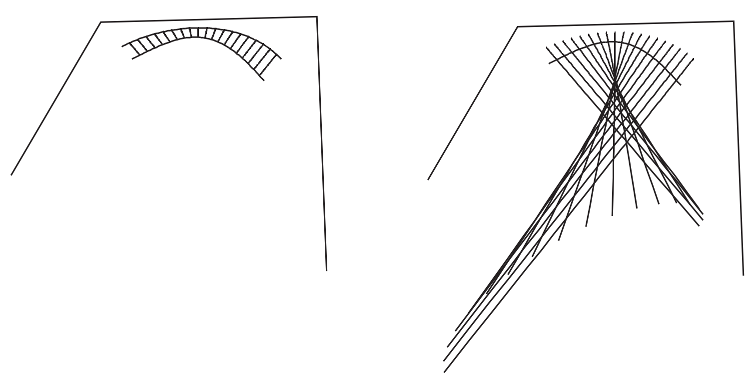

Fig. 8. Relative α-paths of a C-B-spline arc and their extensions

Just as for C-Bézier curves, apart from the case $ratio = 2$ these paths are not lines but can be approximated by lines. The approximate line of the path $\mathbf{s}(\alpha, ratio)$ can be defined by the joint segment of the point $\mathbf{s}(\pi, ratio)$ and $\mathbf{s}(0, ratio)$.

If $\alpha = \pi$ and $t = \pi/ratio$, then we obtain:

$$
\begin{aligned}
\mathbf{B}_0(\pi/ratio,\pi) &= \frac{ratio\sin(\pi/ratio)+\pi-\pi ratio}{-4\pi ratio}\\
\mathbf{B}_1(\pi/ratio,\pi) &= \frac{-ratio\sin(\pi/ratio)+\pi-2\pi ratio}{-4\pi ratio}\\
\mathbf{B}_2(\pi/ratio,\pi) &= \frac{-ratio\sin(\pi/ratio)-\pi-\pi ratio}{-4\pi ratio}\\
\mathbf{B}_3(\pi/ratio,\pi) &= \frac{ratio\sin(\pi/ratio)-\pi}{-4\pi ratio},
\end{aligned}
\tag{5}
$$

while applying the limit $\alpha \to 0$ for equations (4):

$$
\begin{aligned}
\mathbf{B}_{0lim} &= \frac{ratio^3-3ratio^2+3ratio-1}{6ratio^3}\\
\mathbf{B}_{1lim} &= \frac{4ratio^2-6ratio+3}{6ratio^3}\\
\mathbf{B}_{2lim} &= \frac{ratio^3+3ratio^2+3ratio-3}{6ratio^3}\\
\mathbf{B}_{3lim} &= \frac{1}{6ratio^3}.
\end{aligned}
\tag{6}
$$

Connecting the points $\sum_{i=0}^{3} B_i(\pi/ratio,\pi)\mathbf{p}_i$ and $\sum_{i=0}^{3} B_{i\,\mathrm{lim}}(ratio)\mathbf{p}_i$ the result is a family of lines with family parameter *ratio*. The intersection curve of the symmetric lines has the same property as in the C-Bézier case (here we suppose that the control points are coplanar so the intersection curve exists).

Theorem 2 *The intersection curve of the symmetric lines is a straight line segment if* $\mathbf{p}_0\mathbf{p}_3$ *is parallel to* $\mathbf{p}_1\mathbf{p}_2$*. Furthermore, the line segment is just on the line connecting the midpoints of* $\mathbf{p}_0\mathbf{p}_3$ *and* $\mathbf{p}_1\mathbf{p}_2$*.*

Proof: As well as in the C-Bézier situation, the two pairs of endpoints of the symmetric lines are on the curve $\alpha = 0$ and $\alpha = \pi$ respectively. We prove the result in a special case using the coordinate system given in Fig. 9 and compute the two curves $\mathbf{b_{lim}}(ratio) = lim_{\alpha\to 0}\mathbf{b}(\alpha/ratio,\alpha)$ and $\mathbf{b}(\pi/ratio,\pi)$ by equations (5) and (6). Using the abbreviation $r = ratio$ the points are:

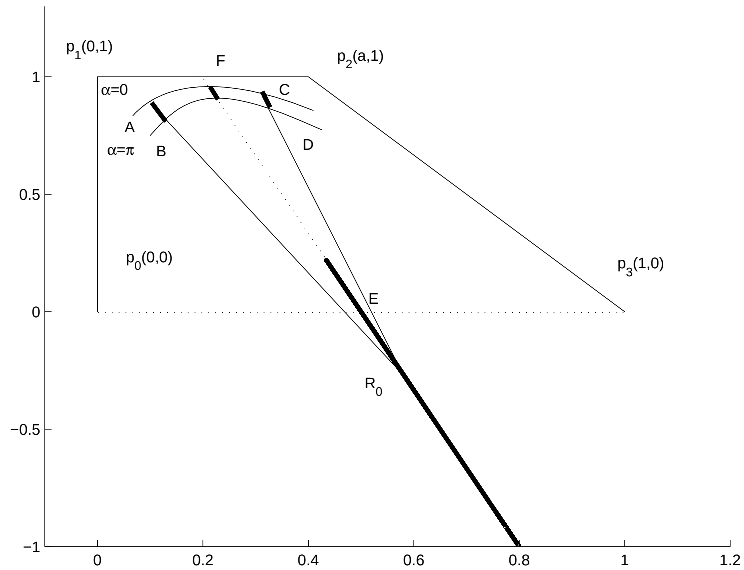

Fig. 9. Control points and the path

$$\mathbf{A} = \sum_{i=0}^{3} B_{ilim}(r)\mathbf{p}_i$$
$$\mathbf{B} = \sum_{i=0}^{3} B_i(\pi/r, \pi)\mathbf{p}_i$$
$$\mathbf{C} = \sum_{i=0}^{3} B_{ilim}(\frac{r}{r-1})\mathbf{p}_i$$
$$\mathbf{D} = \sum_{i=0}^{3} B_i(\pi/\frac{r}{r-1}, \pi)\mathbf{p}_i.$$

The coordinates of these points are as follows:

$$\mathbf{A} = \left(\frac{ar^3 + 3ar^2 + 3ar - 3a + 1}{6r^3}, \frac{5r^2 + 3r - 3}{6r^2}\right)$$
$$\mathbf{B} = \left(\frac{-ar\sin\frac{\pi}{r} - 3a\pi + r\sin\frac{\pi}{r} - \pi}{-4\pi r}, \frac{-2\sin\frac{\pi}{r} - 3\pi}{-4\pi}\right)$$
$$\mathbf{C} = \left(\frac{4ar^3 + r^3 - 3r^2 + 3r + 3a - 1}{6r^3}, \frac{5r^2 + 3r - 3}{6r^2}\right)$$
$$\mathbf{D} = \left(\frac{-ar\sin\frac{\pi}{r} - 3a\pi + 2a\pi r + r\sin\frac{\pi}{r} + \pi - \pi r}{-4\pi r}, \frac{-2\sin\frac{\pi}{r} - 3\pi}{-4\pi}\right).$$

Hence the intersecting point of **AB** and **CD** is $\mathbf{R}(\lambda) = \lambda\mathbf{A} + (1-\lambda)\mathbf{B}$, where:

$$\lambda = \frac{3r^2(a+1)}{(1-3a)(r^2+2r-2)} \quad r \in [1,2].$$

Denote the midpoint of $\mathbf{p}_0\mathbf{p}_3$ by $\mathbf{E}$ and the midpoint of $\mathbf{p}_1\mathbf{p}_2$ by $\mathbf{F}$. It is not difficult to compute that the reciprocal of slope of $\mathbf{EF}$ is

$$\frac{1}{k_{EF}} = \frac{a-1}{2}$$

By the same way, we compute the reciprocal of slope of $\mathbf{ER}(\lambda)$:

$$\frac{1}{k_{\mathbf{ER}(\lambda)}} = \frac{a-1}{2} = \frac{1}{k_{\mathbf{EF}}}.$$

$\square$

The approximate lines of the relative α-paths of C-B-spline curves have a property which has no analogue in the C-Bézier case: for a certain position of control points all the lines are parallel (see Fig. 10).

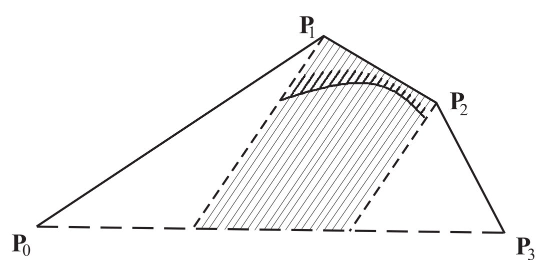

Fig. 10. In a special case paths can be replaced by parallel lines

Theorem 3 *Dividing the line $\mathbf{p}_0\mathbf{p}_3$ into three equal parts by points $\mathbf{q}_1, \mathbf{q}_2$, the approximate lines are parallel if the line $\mathbf{p}_1\mathbf{q}_1$ is parallel to the line $\mathbf{p}_2\mathbf{q}_2$.*

Proof:

Since parallelity is affine invariant, similarly to the previous proofs we use the special case and coordinate system given in Fig. 11 to prove the proposition. If the line $\mathbf{p}_1\mathbf{q}_1$ is parallel to the line $\mathbf{p}_2\mathbf{q}_2$, then

$$\frac{1}{k_{\mathbf{p}_1\mathbf{q}_1}} = -\frac{1}{3} = \frac{1}{k_{\mathbf{p}_2\mathbf{q}_2}} = \frac{a-2/3}{b},$$

hence we obtain $b = 2-3a$. Substitute it into the spline functions (4), we obtain the end points $\mathbf{A}, \mathbf{B}$ of the approximate lines as:

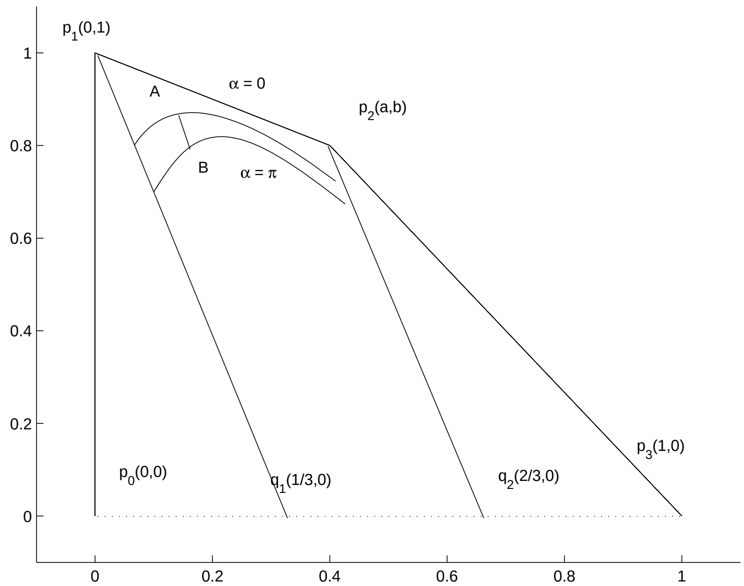

Fig. 11. Control points and the path

$$\mathbf{A} = \left(\frac{ar^3 + 3ar^2 + 3ar - 3a + 1}{6r^3}, \frac{3(2r * 3 + 2r^2 - 1) - 3a(3r^2 + 3r - 3 + r^3)}{6r^3} \right)$$

$$\mathbf{B} = \left(\frac{-a(r \sin\frac{\pi}{r} + \pi + r\pi) + r \sin\frac{\pi}{r} - \pi}{-4\pi r}, \frac{-3r \sin\frac{\pi}{r} - 3\pi - 2r\pi + 3a(r \sin\frac{\pi}{r} + r\pi + \pi)}{-4\pi} \right).$$

Computing the reciprocal of slope of $\mathbf{AB}$, we find that it equals $-1/3$, which completes the proof. □

6 Finding a C-Bézier curve passing through a given point

As a practical application of the theoretical results here we show an example how these approximate lines can be applied in constrained shape modification of C-curves. Let a C-Bézier curve $\mathbf{b}(t, \alpha)$ be given by its control points $\mathbf{p}_i$, $(i = 0, 1, 2, 3)$. Furthermore, let a point $\mathbf{p}$ be given through which the new C-curve should pass. This interpolation problem can simply be solved by the help of approximate lines if the error tolerance is greater than the possible deviation described in Section 3.

Now consider the two curves $\mathbf{b}_{\mathbf{lim}}(t) = lim_{\alpha\to 0}\mathbf{b}(t, \alpha)$ and $\mathbf{b}(t, \pi)$ defined by the original control points $\mathbf{p}_i$. If the given point $\mathbf{p}$ is inside the region defined by these two limit curves, then the modification can be solved by purely altering the parameter α, without modifying the control points, i.e. without modifying the original convex hull of the curve (see Fig.12). At a certain checking point of the computation given below it automatically turns out if $\mathbf{p}$ is inside the restricted area.

We have to find the parameters $\overline{t}$ and $\overline{\alpha}$ for which $\mathbf{b}(\overline{t}, \overline{\alpha}) = \mathbf{p}$ holds. But - due to the trigonometric basis functions - to find this curve or even to find the relative α-path $\mathbf{s}(\overline{\alpha}, \overline{ratio}) = \mathbf{p}$ by direct computation seems to be very complicated if it can be solved at all. We will use a much more easier computation to find the approximate line for which $\mathbf{e}(\overline{s}, \overline{ratio}) = \mathbf{p}$ and then, using the equations

$$s = \left(\frac{\alpha}{\pi}\right)^2$$
$$t = \left(\frac{\alpha}{ratio}\right)$$

one can find the required parameters. At the first step we find the parameter $\overline{ratio}$ for which the line $\mathbf{e}(s, \overline{ratio})$ passes through the given point for some s. This step can be solved numerically as follows: let $ratio$ run from 1 to ∞ and consider the two points of the approximate lines on the limit curves: the points $\sum_{i=0}^{3} Z_i(\pi/ratio, \pi)\mathbf{p}_i$ and $\sum_{i=0}^{3} Z_{i\,\mathrm{lim}}(ratio)\mathbf{p}_i$. Also consider the lines connecting the points $\sum_{i=0}^{3} Z_i(\pi/ratio, \pi)\mathbf{p}_i$ and $\mathbf{p}$. If these two lines are parallel for some parameter $\overline{ratio}$, then the line $\mathbf{e}(s, \overline{ratio})$ passes through $\mathbf{p}$.

At the second step we find $\overline{s}$ by solving

$$\mathbf{p} = \sum_{i=0}^{3}(sZ_i(\pi/\overline{ratio}, \pi) + (1-s)Z_{i\,\mathrm{lim}}(\overline{ratio}))\mathbf{p}_i$$

for s (if $s < 0$ or $s > 1$, then the given point was originally out of the area defined by the limit curves - this is the checking point of the restriction for $\mathbf{p}$ mentioned above). Finally $\mathbf{e}(\overline{s}, \overline{ratio}) = \mathbf{p}$ holds and this yields the equation (with some possible deviation)

$$\mathbf{b}(\overline{t}, \overline{\alpha}) = \mathbf{p}$$

for

$$\overline{t} = \frac{\sqrt{\overline{s}}\pi}{\overline{ratio}}$$
$$\overline{\alpha} = \sqrt{\overline{s}}\pi.$$

Here we remark, that even if the deviation is greater than the predefined error tolerance, the same computation can be executed recursively by substituting one of the limit curves by the curve $\mathbf{b}(t, \overline{\alpha})$ to achieve better result. Since $\mathbf{p}$

is either between $\mathbf{b_{lim}}(t)$ and $\mathbf{b}(t, \overline{\alpha})$ or $\mathbf{b}(t, \pi)$ and $\mathbf{b}(t, \overline{\alpha})$, the new value of the parameter s will be inside the permissible interval $[0, 1]$ in exactly one of the two cases - and this case yields even more precise approximation than the original algorithm.

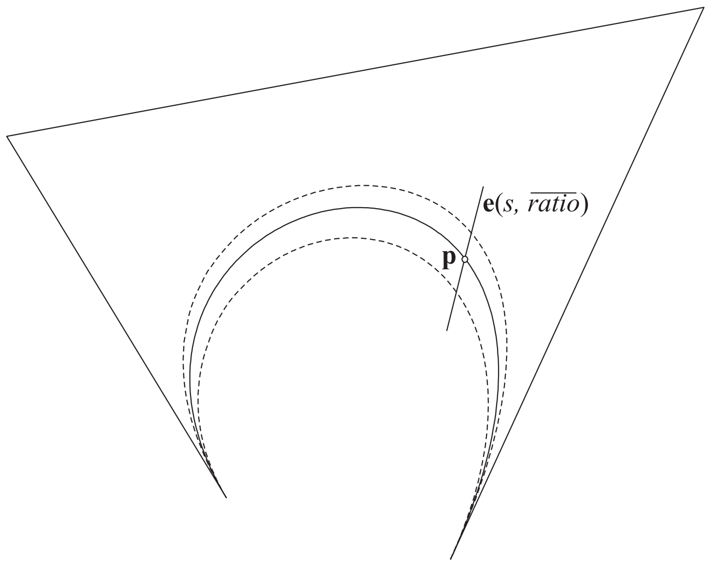

Fig. 12. The two limit curves (dashed line) and the C-Bézier curve passing through the given point $\mathbf{p}$. The approximate line $\mathbf{e}(s, \overline{ratio})$ has been used in the computation.

7 Conclusion and future work

Using C-curves in geometric modeling one has to define their parameter α. The effects of the change of this parameter for the shape of the curve has been discussed and clarified by relative path of points and their approximate lines.

To examine the intersection curve of these lines we assumed that the control points are coplanar. If this does not hold, then the approximate lines form a line surface. In this case the lines all intersect the leg $\mathbf{p}_1\mathbf{p}_2$. Further properties of their surface can be the field of future research as well as the extensions of our results to other types of spline curves.

These results will hopefully lead to further shape modification tools of C-curves by altering their data, including the parameter α.

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List of changes in

”Paths of C-Bézier and C-B-spline curves”

- P5L-3: the barycentric coordinates $((\alpha_0/\pi), 1 - (\alpha_0/\pi))$ have been substituted by $((\alpha_0/\pi)^2, 1 - (\alpha_0/\pi)^2)$. With this new coordinates the standard deviation is far less than before (see below).
- top of P6: for the sake of simplicity the functions E_i were introduced.
- P6: with this new notation the *stdev* function is getting more simple
- P6Fig.4: the standard deviation is much smaller with the new barycentric coordinates
- From the bottom of P6 to Section 4: to provide a reference distance and a geometric meaning of the deviation the difference functions $\triangle_i$ were introduced and it is showed that even by highly overestimating the deviation it is fairly small related to the legs of the control polygon. A new figure (Fig.5) is also inserted to show the graph of the difference functions (this is an answer to the comment of Reviewer #2).
- Section 6 (including Fig.12) is completely new giving an example for practical usage of approximate lines (this is an answer to the comment of Reviewer #1).